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The Structure of Locally Finite Split Lie Algebras

Nina Stumme

Fachbereich Mathematik, Technische Universität Darmstadt, Schloßgartenstraße 7, D-64289 Darmstadt, Germany
E-mail: nstumme@mathematik.tu-darmstadt.de

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If $\mathfrak{g}$ is a split Lie algebra, which means that $\mathfrak{g}$ is a Lie algebra with a root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\alpha}$, then the roots of Δ can be classified into different types: a root $\alpha \in \Delta$ is said to be of nilpotent type if all subalgebras $\mathfrak{g}(x_\alpha, x_{-\alpha}) := \operatorname{span}_{\mathbb{K}}\{x_\alpha, x_{-\alpha}, [x_\alpha, x_{-\alpha}]\}$ for $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ are nilpotent, and of simple type if there exist elements $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $\mathfrak{g}(x_\alpha, x_{-\alpha}) \cong \mathfrak{sl}(2, \mathbb{K})$. A simple root $\alpha \in \Delta$ is called integrable if there exist elements $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $\mathfrak{g}(x_\alpha, x_{-\alpha}) \cong \mathfrak{sl}(2, \mathbb{K})$ and the endomorphisms $\operatorname{ad} x_{\pm\alpha}$ are locally nilpotent (Section I).

The role of integrable roots in split Lie algebras has been investigated by K.-H. Neeb in [Ne98]. One important result of this paper is the Local Finiteness Theorem which states that a split Lie algebra with only integrable roots is locally finite, i.e., the Lie algebra is the direct limit of its finite dimensional subalgebras.

In this paper we focus from the outset on locally finite split Lie algebras. Our objective is to describe the correspondence between the root types of Δ and the structural features of a locally finite split Lie algebra $\mathfrak{g}$.

If $\mathfrak{g}$ is finite dimensional, then $\mathfrak{g}$ has a unique $\mathfrak{h}$-invariant Levi decomposition where the radical as well as the $\mathfrak{h}$-invariant Levi complement can be described in terms of root types (Theorem II.1). One of the main results of this paper is an analog of this statement for locally finite split Lie algebras, saying that a locally finite split Lie algebra $\mathfrak{g}$ has a generalized Levi decomposition. This means that $\mathfrak{g} \cong \mathfrak{r} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$ where $\mathfrak{r}$ is the unique maximal locally solvable ideal of $\mathfrak{g}$, $\mathfrak{s}$ is an $\mathfrak{h}$-invariant semisimple subalgebra of $\mathfrak{g}$ that is generated by the root spaces of integrable roots, and $\mathfrak{a}$ is a subspace of the abelian Lie algebra $\mathfrak{h}$ (Theorem III.16). The existence of a

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Levi decomposition in a certain class of locally finite, not necessarily split, Lie algebras has also been investigated by R. Amayo and I. Stewart in the book [AmSt74] (cf. Remark III.18).

The generalized Levi decomposition of a locally finite split Lie algebra permits us to give a characterization of the semisimple Lie algebras, i.e., the Lie algebras that are the direct sums of simple ideals: We show that a locally finite split Lie algebra is semisimple if and only if it is perfect and has only integrable roots (Theorem III.19).

Similar results are obtained for locally finite split graded Lie algebras, which constitute the framework of [Ne98] and give a slightly more general framework than split Lie algebras (Section IV).

In Sections V and VI we study the structure of the root system Δ of a locally finite split semisimple Lie algebra $\mathfrak{g}$, using the results about general root systems provided in Section I and the structure theorems of Section III. We show that Δ is the directed union of finite root subsystems of semisimple type and, moreover, that Δ is the directed union of finite root systems of simple type if $\mathfrak{g}$ is simple. From the latter we derive that a locally finite split simple Lie algebra is the direct limit of finite dimensional simple subalgebras. Section VI contains the proof of the existence of a generalized base in the root system Δ, which is a linearly independent subset Φ of Δ with $\Delta \subseteq \operatorname{span}_{\mathbb{Z}} \Phi$ (Theorem VI.6). The notion of a generalized base supplies a substitute for a root base of Δ, which only exists if Δ is countable, and is thus an effective tool to study the structure of locally finite split semisimple Lie algebras. Some of the consequences of its existence are given in Sections VII and VIII. In particular, it is shown that every complex locally finite split semisimple Lie algebra has a “compact” real form (Theorem VIII.5).

Generalized bases also play a crucial role in the classification of the locally finite split simple Lie algebras over a field $\mathbb{K}$ of characteristic zero, which is performed in the forthcoming paper [NeSt99] and states that, without any countability assumption, each locally finite, infinite dimensional, split simple Lie algebra is isomorphic to exactly one of the Lie algebras $\mathfrak{sl}(J, \mathbb{K})$, $\mathfrak{o}(J, J, \mathbb{K})$, or $\mathfrak{sp}(J, \mathbb{K})$ where J is an infinite set whose cardinality equals the dimension of $\mathfrak{g}$. This classification leads, furthermore, to the classification of the simple L^*-algebras, which are simple involutive Banach–Lie algebras whose underlying vector space is a Hilbert space with a contravariant scalar product. This application uses an old result of Schue saying that every simple L^*-algebra contains a dense locally finite split simple Lie algebra [Sch61]. A classification of the separable L^*-algebras was given by J. Schue in [Sch60] and [Sch61].

Simple locally finite Lie algebras have been studied in various contexts. Y. Bakhturin and H. Strade [BaSt95a, BaSt95b] investigate general properties of locally finite simple Lie algebras. They describe examples of locally finite simple Lie algebras which are not the directed union of a set of finite dimensional simple subalgebras and, hence, in view of our results, have no root decomposition. Y. Bakhturin and G. Benkart [BaBe97] study generalized highest weight representations of locally finite, not necessarily split, Lie algebras, and the work of I. Dimitrov and I. Penkov [DiPe98] is concerned with the structure of weight modules of locally finite split simple Lie algebras. Furthermore unitary highest weight representations of classical locally finite split simple Lie algebras are studied in an analytical context by K.-H. Neeb and B. Ørsted [Ne97, NeØr98].

I. TEST ALGEBRAS AND ROOT TYPES

In this section we distinguish the various root types in the root system of a split Lie algebra, taking particular notice of the integrable roots, which influence the structure of the root decomposition strongly.

Throughout this paper $\mathbb{K}$ denotes a field of characteristic zero and $\mathfrak{g}$ a Lie algebra over $\mathbb{K}$.

DEFINITION I.1. We call an abelian subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ a *splitting Cartan subalgebra* if $\mathfrak{h}$ is maximal abelian and the endomorphisms $\operatorname{ad} h$ for $h \in \mathfrak{h}$ are simultaneously diagonalizable. If $\mathfrak{g}$ contains a splitting Cartan subalgebra $\mathfrak{h}$, then the pair $(\mathfrak{g}, \mathfrak{h})$ (or, simply, $\mathfrak{g}$) is called a *split Lie algebra*. This means that we have a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})[h, x] = \alpha(h)x\}$ for a linear functional $\alpha \in \mathfrak{h}^*$ and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \left\{\alpha \in \mathfrak{h}^* \backslash \{0\} : \mathfrak{g}^\alpha \neq \{0\}\right\}$$

is the corresponding root system. The subspaces $\mathfrak{g}^\alpha$ for $\alpha \in \Delta$ are called *root spaces* and its elements are called *root vectors*.

In the following $(\mathfrak{g}, \mathfrak{h})$ denotes a split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition.

LEMMA I.2. *For non-zero root vectors $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ the subalgebra*

$$\mathfrak{g}(x_\alpha, x_{-\alpha}) := \operatorname{span}_{\mathbb{K}}\{x_\alpha, x_{-\alpha}, [x_\alpha, x_{-\alpha}]\}$$

is of one of the following types:

(A) *If $[x_\alpha, x_{-\alpha}] = 0$, then $\mathfrak{g}(x_\alpha, x_{-\alpha})$ is two dimensional abelian. We say that $\mathfrak{g}(x_\alpha, x_{-\alpha})$ is of abelian type.*

(N) *If* $[x_\alpha, x_{-\alpha}] \neq 0$ *but* $\alpha([x_\alpha, x_{-\alpha}]) = 0$, *then* $\mathfrak{g}(x_\alpha, x_{-\alpha})$ *is a three dimensional Heisenberg algebra. We say that* $\mathfrak{g}(x_\alpha, x_{-\alpha})$ *is of nilpotent type.*

(S) *If* $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$, *then* $\mathfrak{g}(x_\alpha, x_{-\alpha}) \cong \mathfrak{sl}(2, \mathbb{K})$. *We say that* $\mathfrak{g}(x_\alpha, x_{-\alpha})$ *is of simple type.*

Proof. It is clear that the space $\mathfrak{a} := \mathfrak{g}(x_\alpha, x_{-\alpha})$ is a subalgebra of $\mathfrak{g}$ because $[x_\alpha, x_{-\alpha}] \in \mathfrak{h}$. Let $z := [x_\alpha, x_{-\alpha}]$. If $z = 0$, then $\mathfrak{a}$ is two dimensional abelian. If $z \neq 0$ but $\alpha(z) = 0$, then $z \in \mathfrak{z}(\mathfrak{a})$, showing that $\mathfrak{a}$ is a three dimensional Heisenberg algebra. If $\alpha(z) \neq 0$, then putting

$$h := \frac{2}{\alpha([x_\alpha, x_{-\alpha}])}[x_\alpha, x_{-\alpha}], \qquad e := \frac{2}{\alpha([x_\alpha, x_{-\alpha}])}x_\alpha$$

and $f = x_{-\alpha}$, we obtain $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Hence $\mathfrak{a} \cong \mathfrak{sl}(2, \mathbb{K})$. ∎

Definition I.3. (a) For a root $\alpha \in \Delta$ the subalgebras $\mathfrak{g}(x_\alpha, x_{-\alpha})$ for $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ are called *test algebras* associated to α.

(b) We say that a root $\alpha \in \Delta$ is of *nilpotent type* if all test algebras associated to α are of abelian or nilpotent type. Note that a root $\alpha \in \Delta$ with $-\alpha \notin \Delta$ is of nilpotent type. We call a root $\alpha \in \Delta$ of *simple type* if there exists an associated test algebra of simple type. A root $\alpha \in \Delta$ of *simple type* is called *integrable* if there exists an associated test algebra $\mathfrak{g}(x_\alpha, x_{-\alpha})$ of simple type such that the endomorphisms $\operatorname{ad} x_{\pm\alpha}$ are locally nilpotent. We write Δ_n for the set of roots of nilpotent type, Δ_s for the set of roots of simple type and Δ_i for the set of integrable roots. Observe that $\Delta = \Delta_n \dot{\cup} \Delta_s$ and that $\Delta_s = -\Delta_s$ and $\Delta_i = -\Delta_i$ follow from the symmetry in the definition of the root types. ∎

To give some examples for the various root types, and to illustrate some notions introduced later in this paper, we briefly discuss certain typical examples of Lie algebras with root decompositions.

Example I.4. (a) Consider the following endomorphisms of the algebra $S = \mathbb{K}[X_n, n \in \mathbb{N}]$: the left multiplication $l(X_n)$ by X_n and the partial derivative $\partial/\partial X_n$ with respect to X_n for $n \in \mathbb{N}$ and the identity operator $\mathbf{1}$. The linear span $\mathfrak{g}$ of these endomorphisms is a Heisenberg algebra with the Lie bracket $[\partial/\partial X_n, l(X_m)] = \delta_{nm}\mathbf{1}$ for $n, m \in \mathbb{N}$ and all other brackets equal to zero. Let D be the endomorphism of $\mathfrak{g}$ given by

$$D(l(X_n)) = n\, l(X_n), \qquad D\left(\frac{\partial}{\partial X_n}\right) = -n\frac{\partial}{\partial X_n} \qquad \text{and} \qquad D(\mathbf{1}) = 0.$$

Then D is a derivation of $\mathfrak{g}$. Extending $\mathfrak{g}$ with this derivation, we obtain a Lie algebra $\mathfrak{g}_\mathfrak{d} = \mathfrak{g} \rtimes \mathbb{K}D$ with the additional brackets $[D, x] = D(x)$ for $x \in \mathfrak{g}$, which is a generalized oscillator algebra. Moreover $\mathfrak{g}_\mathfrak{d}$ is a split Lie algebra, and its subalgebra $\mathfrak{h} = \mathbb{K}\mathbf{1} + \mathbb{K}D$ is a splitting Cartan subalgebra. With $\lambda_1\colon \mathfrak{h} \to \mathbb{K}$ given by $\lambda_1(\mathbf{1}) = 0$ and $\lambda_1(D) = 1$ we have

$$\Delta = \Delta(\mathfrak{g}_{\mathfrak{d}}, \mathfrak{h}) = \{n\lambda_1 : n \in \mathbb{Z}\backslash\{0\}\},$$

and the corresponding root spaces are $(\mathfrak{g}_{\mathfrak{d}})^{n\lambda_1} = \mathbb{K}l(X_n)$ and $(\mathfrak{g}_{\mathfrak{d}})^{-n\lambda_1} = \mathbb{K}(\partial/\partial X_n)$. All roots of Δ are of nilpotent type.

(b) The Virasoro algebra, which has a realisation as the vector space $\mathfrak{g}$ with basis $\{L_n, c : n \in \mathbb{Z}\}$ and Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}c \qquad \text{and} \qquad [L_m, c] = 0$$

for n, $m \in \mathbb{Z}$, is a split Lie algebra, and its subalgebra $\mathfrak{h} = \mathbb{K}L_0 + \mathbb{K}c$ is a splitting Cartan subalgebra. If we define $\lambda_1\colon \mathfrak{h} \to \mathbb{K}$ by $\lambda_1(L_0) = -1$ and $\lambda_1(c) = 0$ for $n \in \mathbb{Z}\backslash\{0\}$, then the root system is $\Delta(\mathfrak{g}, \mathfrak{h}) = \{n\lambda_1 : n \in \mathbb{Z}\backslash\{0\}\}$ and the root spaces are $\mathfrak{g}^{n\lambda_1} = \mathbb{K}L_n$. The roots of $\Delta(\mathfrak{g}, \mathfrak{h})$ are of simple type because $\mathfrak{g}(L_n, L_{-n}) = \operatorname{span}_{\mathbb{K}}\{L_n, L_{-n}, 2nL_0 + \frac{1}{12}(n^3 - n)c\} \cong \mathfrak{sl}(2, \mathbb{K})$, but they are not integrable because $(\operatorname{ad} L_n)^k \cdot L_{m+n} \neq 0$ for $n \in \mathbb{N}$ and all $k \in \mathbb{N}$, showing that $\mathfrak{g}(L_n, L_{-n})$ does not act in a locally finite fashion.

(c) For a set J denote by $\mathfrak{gl}(J, \mathbb{K})$ the set of $J \times J$-matrices with finitely many non-zero entries, where a $J \times J$-matrix is considered as a map $J \times J \to \mathbb{K}$. We write

$$E_{jk}\colon J \times J \to \mathbb{K}, \qquad (l, m) \mapsto \delta_{jl}\delta_{km}$$

for the unit matrix with entry 1 at the coordinates (j, k) and 0 elsewhere. The Lie algebra $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{K})$, endowed with the commutator bracket, has a root decomposition with respect to the subalgebra $\mathfrak{h} = \operatorname{span}_{\mathbb{K}}\{E_{jj} j \in J\}$ of diagonal matrices. If we set $\varepsilon_j\colon \mathfrak{h} \to \mathbb{K}$, $E_{kk} \mapsto \delta_{jk}$, then the corresponding root system is $\Delta = \{\varepsilon_j - \varepsilon_k : j, k \in J, j \neq k\}$ and the root spaces are $\mathfrak{g}^{\varepsilon_j - \varepsilon_k} = \mathbb{K}E_{jk}$. It is easy to check that all roots of Δ are integrable. ∎

LEMMA I.5. *Let $\alpha \in \Delta$ and $\mathfrak{g}(x_\alpha, x_{-\alpha})$ be a test algebra associated to α such that the endomorphisms* $\operatorname{ad} x_{\pm\alpha}$ *are locally nilpotent. Then $\mathfrak{g}$ is a locally finite $\mathfrak{g}(x_\alpha, x_{-\alpha})$-module with respect to the adjoint representation.*

Proof. This is shown using the same arguments as for $\mathfrak{sl}(2, \mathbb{K})$ (cf. [MoPi95, Proposition 2.4.7]). ∎

Naturally, one expects to determine global properties of the Lie algebra $\mathfrak{g}$ in terms of its test algebras and root types. Of particular importance is the role of the integrable roots: If $\alpha \in \Delta_i$, then the preceding lemma shows that there exists a test algebra $\mathfrak{g}(x_\alpha, x_{-\alpha}) \cong \mathfrak{sl}(2, \mathbb{K})$ such that $\mathfrak{g}$ is a locally finite $\mathfrak{g}(x_\alpha, x_{-\alpha})$-module. Therefore we can apply the representation theory of $\mathfrak{sl}(2, \mathbb{K})$ to the finite dimensional submodules of $\mathfrak{g}$ and from this gain information about the root system Δ and the structure of the root decomposition of $\mathfrak{g}$.

PROPOSITION I.6. *For $\alpha \in \Delta_i$ the following assertions hold*:

(i) $\dim \mathfrak{g}^\alpha = 1$.

(ii) *There exists a unique element* $\check{\alpha} \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ *with* $\alpha(\check{\alpha}) = 2$.

(iii) *For each root* $\beta \in \Delta$ *we have* $\beta(\check{\alpha}) \in \mathbb{Z}$.

(iv) $\mathbb{K}\alpha \cap \Delta \subseteq \{\pm\alpha\} \cup (\frac{1}{2} + \mathbb{Z})\alpha$.

(v) $\mathbb{K}\alpha \cap \Delta_i = \{\pm\alpha\}$.

Proof. Since α is integrable, we find $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $\mathfrak{a} := \mathfrak{g}(x_\alpha, x_{-\alpha}) \cong \mathfrak{sl}(2, \mathbb{K})$ and $\operatorname{ad} x_{\pm\alpha}$ are locally nilpotent. We may w.l.o.g. assume that $\alpha([x_\alpha, x_{-\alpha}]) = 2$ and put $h := [x_\alpha, x_{-\alpha}]$, $e := x_\alpha$, and $f := x_{-\alpha}$.

(i) We consider the $\mathfrak{a}$-submodule

$$V := \mathbb{K}f + \mathfrak{h} + \sum_{n=1}^{\infty} \mathfrak{g}^{n\alpha} \subseteq \mathfrak{g}.$$

As a submodule of a locally finite module, V is locally finite. In particular V is a sum of finite dimensional simple $\mathfrak{a}$-submodules by Weyl's Theorem. Hence the representation theory of $\mathfrak{sl}(2, \mathbb{K})$ implies that the set $\mathcal{P}_V(h)$ of h-eigenvalues on V is symmetric with $\dim V^\mu(h) = \dim V^{-\mu}(h)$ for each $\mu \in \mathbb{K}$. Now $V^{-2}(h) = \mathbb{K}f$ implies that $\dim V^2(h) = \dim \mathfrak{g}^\alpha = 1$ and furthermore that $\dim V^{2n}(h) = \dim \mathfrak{g}^{n\alpha} = \{0\}$ for $n > 1$. Likewise $\mathfrak{g}^{-n\alpha} = \{0\}$ for $n > 1$.

(ii) Since both spaces $\mathfrak{g}^{\pm\alpha}$ are one dimensional and do not commute, the space $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ is one dimensional. Hence the element $\check{\alpha} := h$ is uniquely determined by $\alpha(\check{\alpha}) = 2$.

(iii) Since $\mathfrak{g}$ is a locally finite $\mathfrak{a}$ module and $\beta(\check{\alpha})$ is the eigenvalue of $\check{\alpha}$ on the root space $\mathfrak{g}^\beta$, this is a consequence of the finite dimensional representation theory of $\mathfrak{sl}(2, \mathbb{K})$.

(iv) Let $\beta = c\alpha \in \Delta$ with $c \in \mathbb{K}$. Then (iii) implies that $c \in \frac{1}{2}\mathbb{Z}$. If $c \in \mathbb{Z}$, then $c = \pm 1$ is a consequence of the proof of (i).

(v) If, in addition, β is integrable, then we also have $\frac{1}{c} \in \frac{1}{2}\mathbb{Z}$ and thus $\frac{4}{2c} \in \mathbb{Z}$. Since $2c$ divides 4 it equals 1, 2, or 4, so that we may have $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. The case $c = \pm 2$ is ruled out by (iv) and likewise the case $c = \pm\frac{1}{2}$. Hence $c = \pm 1$. ∎

We have seen that for integrable roots the root spaces $\mathfrak{g}^{\pm\alpha}$ are one dimensional, showing that the test algebras $\mathfrak{g}(x_\alpha, x_{-\alpha})$ do not depend on the choice of $x_{\pm\alpha}$. We write

$$\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + \mathbb{K}\check{\alpha}$$

for the corresponding test algebra. The element $\check{\alpha} \in \mathfrak{h}$ is called the associated *coroot*.

The following proposition describes the consequences of the finite dimensional representation theory of $\mathfrak{sl}(2,\mathbb{K})$ for the $\mathfrak{g}(\alpha)$-submodules $\sum_k \mathfrak{g}^{\beta+k\alpha}$ of $\mathfrak{g}$.

PROPOSITION I.7. *For $\alpha \in \Delta_i$ and $\beta \in \Delta \setminus \{\pm\alpha\}$ the following assertions hold*:

(i) *The set $\{k \in \mathbb{Z}\colon \beta + k\alpha \in \Delta\}$ is an interval in $\mathbb{Z}$.*

(ii) *If this interval is bounded and equal to $[-p, q] \cap \mathbb{Z}$, then $p, q \in \mathbb{N}_0$ and $p - q = \beta(\check{\alpha})$.*

(iii) *If $\beta(\check{\alpha}) < 0$, then $\beta + \alpha \in \Delta$.*

(iv) *If $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \{0\}$, then $\beta(\check{\alpha}) \geq 0$.*

(v) *If $\dim \mathfrak{g}^\beta = 1$, then $\mathfrak{g}^\beta$ generates a finite dimensional simple $\mathfrak{g}(\alpha)$-module intersecting every root space in the α-string through β. In particular, if α, β and $\alpha+\beta$ are integrable roots, then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$.*

(vi) *If $\dim \mathfrak{g}^\beta = 1$, then for $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ and $z_\beta \in \mathfrak{g}^\beta$ we have*

$$\big[x_{-\alpha}, [x_\alpha, z_\beta]\big] = \frac{\alpha([x_\alpha, x_{-\alpha}])}{2} q(p+1) z_\beta.$$

Proof. (i)–(iv) follow the standard proof for the corresponding facts for Kac–Moody algebras and finite dimensional semisimple Lie algebras (cf. [Ka90, Propostion 3.6]). All these properties are direct consequences of the fact that $\sum_{k\in\mathbb{Z}} \mathfrak{g}^{\beta+k\alpha}$ is a locally finite module of $\mathfrak{g}(\alpha)$.

(v) Denote by $W \subseteq V := \sum_{k\in\mathbb{Z}} \mathfrak{g}^{\beta+k\alpha}$ the $\mathfrak{g}(\alpha)$-submodule generated by the one dimensional root space $\mathfrak{g}^\beta$. Then W is a finite dimensional module of the Lie algebra $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2,\mathbb{K})$, and since the $\beta(\check{\alpha})$-eigenspace $\mathfrak{g}^\beta$ for $\check{\alpha}$ on W is one dimensional, the representation theory of $\mathfrak{sl}(2,\mathbb{K})$ implies that W is a simple $\mathfrak{g}(\alpha)$-module.

The subspace V is a locally finite $\mathfrak{g}(\alpha)$-module and therefore a sum of finite dimensional $\mathfrak{g}(\alpha)$-submodules by Weyl's Theorem. Since each $\mathfrak{g}(\alpha)$-submodule of V is adapted to the decomposition of V in the $\check{\alpha}$-eigenspaces $\mathfrak{g}^{\beta+k\alpha}$, $k \in \mathbb{Z}$, there exists for each root $\beta + l\alpha \in \Delta$ a finite dimensional simple $\mathfrak{g}(\alpha)$-submodule W' with $W' \cap \mathfrak{g}^{\beta+l\alpha} \neq \{0\}$. If $\beta + l\alpha \in \Delta$ with $|(\beta+l\alpha)(\check{\alpha})| \geq |\beta(\check{\alpha})|$, then the representation theory of $\mathfrak{sl}(2,\mathbb{K})$ implies that all numbers $(\beta+k\alpha)(\check{\alpha})$ with $|(\beta+k\alpha)(\check{\alpha})| \leq |(\beta+l\alpha)(\check{\alpha})|$ are $\check{\alpha}$-eigenvalues of W', showing that W' intersects the corresponding eigenspaces $\mathfrak{g}^{\beta+k\alpha}$ non-trivially. In particular, we have $\mathfrak{g}^\beta \subseteq W'$ and hence $W' = W$. This implies the first statement because $|(\beta+l\alpha)(\check{\alpha})| \leq |\beta(\check{\alpha})|$ trivially implies that $W \cap \mathfrak{g}^{\beta+l\alpha} \neq \{0\}$.

If β is an integrable root, then $\dim \mathfrak{g}^\beta = 1$ and the $\mathfrak{g}(\alpha)$-module W generated by $\mathfrak{g}^\beta$ intersects the one dimensional root space $\mathfrak{g}^{\alpha+\beta}$ non-trivially and thus contains it. This implies that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^\alpha \cdot W^{\beta(\check{\alpha})} = W^{(\alpha+\beta)(\check{\alpha})} = \mathfrak{g}^{\alpha+\beta}$ where $W^{\beta(\check{\alpha})}$ and $W^{(\alpha+\beta)(\check{\alpha})}$ denote $\check{\alpha}$-eigenspaces of W.

(vi) Normalizing $x_{\pm\alpha}$ we may w.l.o.g. assume that $[x_\alpha, x_{-\alpha}] = \check{\alpha}$, so that $\alpha([x_\alpha, x_{-\alpha}]) = 2$. Then the statement follows as in [Hum72, Lemma 25.2]. ∎

The following lemmas, which generalize [BoSi49, Theorem 2.1], will be used in Section V where they will be applied to the root system of a locally finite split semisimple Lie algebra.

LEMMA I.8. *Let $M \subseteq \Delta_i$ be a finite subset and*

$$\beta = \sum_{\alpha \in M} n_\alpha \alpha \in \Delta_i \backslash M$$

such that $n_\alpha \in \mathbb{N}_0$ for all $\alpha \in M$. Then there exists a root $\alpha_0 \in M$ such that $\alpha_0 - \beta \in \Delta$.

Proof. Suppose, on the contrary, that $\alpha - \beta \notin \Delta$ for all $\alpha \in M$. Then Proposition I.7(iii) implies that $\alpha(\check{\beta}) \leq 0$ for all $\alpha \in M$. Therefore we get $2 = \beta(\check{\beta}) = \sum_{\alpha \in M} n_\alpha \alpha(\check{\beta}) \leq 0$, which is absurd. ∎

LEMMA I.9. *Suppose that all roots in Δ are integrable. Let $M \subseteq \Delta$ be a finite subset and $\beta = \sum_{\alpha \in M} n_\alpha \alpha \in \Delta$ where $n_\alpha \in \mathbb{N}_0$. Then we can write $\beta = \sum_{j=1}^{n} \alpha_j$ where $\alpha_j \in M$ such that $\sum_{j=1}^{k} \alpha_j \in \Delta$ for $k = 1, \ldots, n$.*

Proof. Using Lemma I.8 and $\Delta = -\Delta$, the statement can be proved by induction over $\mathrm{ht}\beta := \sum_{\alpha \in M} n_\alpha$. ∎

II. THE LEVI DECOMPOSITION OF A FINITE DIMENSIONAL SPLIT LIE ALGEBRA

In this short section we explain how the different types of roots introduced in Definition I.3 are related to the structure of a split Lie algebra if the Lie algebra is finite dimensional. For similar results we refer to [Ne99, Chapter 7].

THEOREM II.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a finite dimensional split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition. Then $\mathfrak{g}$ contains a unique $\mathfrak{h}$-invariant Levi complement $\mathfrak{s}$, which means that $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ where $\mathfrak{r} := \mathrm{rad}\,\mathfrak{g}$ denotes the radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a Levi subalgebra with $[\mathfrak{h}, \mathfrak{s}] \subseteq \mathfrak{s}$. Moreover, we have*

$$\mathfrak{s} = \mathrm{span}_{\mathbb{K}} \check{\Delta}_i + \sum_{\alpha \in \Delta_i} \mathfrak{g}^\alpha \qquad \text{and} \qquad \mathfrak{r} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{s}) + \sum_{\alpha \in \Delta_n} \mathfrak{g}^\alpha.$$

Proof. First we show that $\mathfrak{s} = \mathrm{span}_{\mathbb{K}} \check{\Delta}_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$ is a Levi complement in $\mathfrak{g}$. Since the radical $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, it is $\mathfrak{h}$-invariant, thus adapted to the root decomposition of $\mathfrak{g}$, and therefore can be written as

$$\mathfrak{r} = (\mathfrak{h}\cap\mathfrak{r}) + \sum_{\alpha\in\Delta}(\mathfrak{g}^\alpha\cap\mathfrak{r}).$$

Let $q\colon \mathfrak{g}\to\mathfrak{g}/\mathfrak{r}$ be the quotient homomorphism and set $\Delta_1 := \{\alpha\in\Delta : \mathfrak{g}^\alpha \not\subseteq \mathfrak{r}\}$. Then $q(\mathfrak{g}) = q(\mathfrak{h}) + \sum_{\alpha\in\Delta_1} q(\mathfrak{g}^\alpha)$, where $q(\mathfrak{h})$ is a splitting Cartan subalgebra of $q(\mathfrak{g})$. Every root $\alpha\in\Delta_1$ vanishes on $\mathfrak{h}\cap\mathfrak{r}$ because otherwise we get $\mathfrak{g}^\alpha = \alpha(\mathfrak{h}\cap\mathfrak{r})\,\mathfrak{g}^\alpha = [\mathfrak{h}\cap\mathfrak{r},\mathfrak{g}^\alpha]\subseteq\mathfrak{r}$, so that α induces a linear functional $\alpha'\colon q(\mathfrak{h})\to\mathbb{K}$ by $\alpha'(q(h)) = \alpha(h)$. With this notation we have $\Delta(q(\mathfrak{g}), q(\mathfrak{h})) = \{\alpha' : \alpha\in\Delta_1\}$ and $q(\mathfrak{g}^\alpha) = q(\mathfrak{g})^{\alpha'}$ for $\alpha'\in\Delta(q(\mathfrak{g}), q(\mathfrak{h}))$. Since $q(\mathfrak{g})$ is semisimple, there exist for each $\alpha\in\Delta_1$ root vectors $q(x_{\pm\alpha})\in q(\mathfrak{g})^{\pm\alpha'}$ such that $0\neq\alpha'([q(x_\alpha), q(x_{-\alpha})]) = \alpha([x_\alpha, x_{-\alpha}])$, implying that $\alpha\in\Delta_i$. Therefore we obtain that

$$\mathfrak{g}/\mathfrak{r} = q(\mathfrak{g}) = \langle\langle q(\mathfrak{g}^\alpha) : \alpha\in\Delta_1\rangle\rangle = q(\mathfrak{s})$$

and hence that $\mathfrak{g} = \mathfrak{r}+\mathfrak{s}$. In order to see that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ it suffices to show that for $\alpha,\beta\in\Delta_i$ with $\alpha\neq\beta$ and $\alpha+\beta\in\Delta$ we have $[\mathfrak{g}^\alpha,\mathfrak{g}^\beta]\subseteq\mathfrak{s}$. If this is not the case, then $\{0\}\neq[\mathfrak{g}^\alpha,\mathfrak{g}^\beta]\subseteq\mathfrak{r}$. In particular, if W denotes the $\mathfrak{g}(\alpha)$-submodule of $\mathfrak{g}$ generated by $\mathfrak{g}^\beta$, then $W\cap\mathfrak{r}$ is a non-zero submodule of W. Since W is simple by Proposition I.7(v), this entails $\mathfrak{g}^\beta\subseteq W\subseteq\mathfrak{r}$. We conclude that $\mathfrak{sl}(2,\mathbb{K})\cong\mathfrak{g}(\beta)\subseteq\mathfrak{r}$, contradicting $\beta\in\Delta_i$. If $\mathfrak{r}_\mathfrak{s}\subseteq\mathfrak{s}$ denotes the solvable radical of $\mathfrak{s}$, then $\mathfrak{r}_\mathfrak{s}\subseteq\ker(q\,|_{\mathfrak{s}}) = \mathfrak{r}\cap\mathfrak{s}$. Since $\mathfrak{r}\cap\mathfrak{s}$ is an $\mathfrak{h}$-invariant ideal of $\mathfrak{s}$ not containing any root space $\mathfrak{g}^\alpha$, $\alpha\in\Delta_i$, we have $\mathfrak{r}\cap\mathfrak{s}\subseteq\mathfrak{h}\cap\mathfrak{s}$ and therefore $\mathfrak{r}\cap\mathfrak{s}\subseteq\mathfrak{z}(\mathfrak{s})$. Thus $\mathfrak{r}_\mathfrak{s}$ is central in $\mathfrak{s}$, showing that $\mathfrak{s}$ is reductive. On the other hand the definition of $\mathfrak{s}$ shows that it is perfect; thus $\mathfrak{s}$ is semisimple and $\mathfrak{r}\cap\mathfrak{s} = \{0\}$. Hence $\mathfrak{g} = \mathfrak{r}\rtimes\mathfrak{s}$, i.e., $\mathfrak{s}$ is a Levi complement of $\mathfrak{g}$. Moreover, we derive that $\mathfrak{r} = (\mathfrak{h}\cap\mathfrak{r}) + \sum_{\alpha\in\Delta_n}\mathfrak{g}^\alpha$ because all simple roots of Δ are integrable and that $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{r}) + (\mathfrak{h}\cap\mathfrak{s})$.

We have seen above that for $\alpha\in\Delta_i = \Delta_1$ we have $\alpha(\mathfrak{h}\cap\mathfrak{r}) = \{0\}$. Therefore $\mathfrak{h}\cap\mathfrak{r}\subseteq\mathfrak{z}_\mathfrak{h}(\mathfrak{s})$ and hence $\mathfrak{z}_\mathfrak{h}(\mathfrak{s}) = (\mathfrak{h}\cap\mathfrak{r}) + (\mathfrak{z}_\mathfrak{h}(\mathfrak{s})\cap\mathfrak{s})$. But $\mathfrak{z}_\mathfrak{h}(\mathfrak{s})\cap\mathfrak{s}\subseteq\mathfrak{z}(\mathfrak{s}) = \{0\}$, showing that $\mathfrak{r} = \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) + \sum_{\alpha\in\Delta_n}\mathfrak{g}^\alpha$.

To see that $\mathfrak{h}$-invariant Levi complements are unique, let $\mathfrak{s}'\subseteq\mathfrak{g}$ be an $\mathfrak{h}$-invariant Levi complement. Then $\mathfrak{s}'$ is adapted to the root decomposition of $\mathfrak{g}$ and satisfies $\mathfrak{g} = \mathfrak{r}\rtimes\mathfrak{s}'$. Therefore $\mathfrak{s}'$ contains all root spaces of integrable roots, implying that $\mathfrak{s}\subseteq\mathfrak{s}'$ and therefore that $\mathfrak{s} = \mathfrak{s}'$. ∎

III. THE GENERALIZED LEVI DECOMPOSITION OF A LOCALLY FINITE SPLIT LIE ALGEBRA

This section contains two central results. We prove the existence of a generalized Levi decomposition for a locally finite split Lie algebra and characterize the locally finite split semisimple Lie algebras. Here semisimple means that the Lie algebra is a direct sum of simple ideals.

As a technical mean we use separated subalgebras, which are subalgebras of a split Lie algebra that are in some strong sense adapted to the root decomposition. We will apply Theorem II.1 to suitable finite dimensional separated subalgebras of a locally finite split Lie algebra and thus gain information about the structure of the Lie algebra in terms of its root types.

DEFINITION III.1. A Lie algebra $\mathfrak{g}$ is called *locally finite dimensional* or simply *locally finite* if every finite dimensional subset of $\mathfrak{g}$ is contained or equivalently generates a finite dimensional subalgebra of $\mathfrak{g}$.

Note that a Lie algebra is locally finite if and only if it is the directed union of its finite dimensional subalgebras.

EXAMPLE III.2. The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{\mathfrak{d}}$ of Example I.4(a) and the Lie algebra $\mathfrak{gl}(J, \mathbb{K})$ of Example I.4(c) are locally finite.

In the following $(\mathfrak{g}, \mathfrak{h})$ denotes a locally finite split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition. Note that all simple roots of Δ are integrable.

LEMMA III.3. *Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}$ that is adapted to the root decomposition*, *i.e.*,

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\alpha\in\Delta_0} (\mathfrak{g}^\alpha \cap \mathfrak{g}_0),$$

where $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $\Delta_0 = \{\alpha \in \Delta : \mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq \{0\}\}$, and let $r\colon \operatorname{span}_{\mathbb{K}} \Delta_0 \to (\mathfrak{h}_0)^*$, $r(\alpha) = \alpha|_{\mathfrak{h}_0}$. Then $\mathfrak{g}_0$ has a weight decomposition

$$\mathfrak{g}_0 = (\mathfrak{g}_0)^0 + \sum_{\beta\in\Delta(\mathfrak{g}_0,\, \mathfrak{h}_0)} (\mathfrak{g}_0)^\beta$$

with respect to $\mathfrak{h}_0$ where $(\mathfrak{g}_0)^0 = \mathfrak{h}_0 + \sum_{r(\alpha)=0} \mathfrak{g}^\alpha$, the weight spaces are $(\mathfrak{g}_0)^\beta = \sum_{\alpha\in\Delta_\beta} (\mathfrak{g}^\alpha \cap \mathfrak{g}_0)$ with $\Delta_\beta = \{\alpha \in \Delta : r(\alpha) = \beta\}$ for $\beta \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)$, and the root system is $\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = r(\Delta_0)\backslash\{0\}$.

Proof. This follows from an easy calculation. ∎

DEFINITION III.4. A subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ that is adapted to the root decomposition is called *separated* if, in the terminology of Lemma III.3, $\mathfrak{h}_0$ is maximal abelian in $\mathfrak{g}_0$ and $\mathfrak{g}^\alpha \cap \mathfrak{g}_0 = (\mathfrak{g}_0)^{r(\alpha)}$ for all $\alpha \in \Delta_0$. This means that $\mathfrak{h}_0$ is a splitting Cartan subalgebra of $\mathfrak{g}_0$ which separates the points of $\Delta_0 \cup \{0\}$.

LEMMA III.5. (a) *Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}$ that is generated by a finite dimensional $\mathfrak{h}$-invariant subspace $\mathfrak{e}$ of $\mathfrak{g}$. Then $\mathfrak{g}_0$ is finite dimensional and* *adapted to the root decomposition of* $\mathfrak{g}$, *i.e.*, $\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\alpha\in\Delta_0}(\mathfrak{g}^\alpha \cap \mathfrak{g}_0)$ *where* $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ *and* $\Delta_0 = \{\alpha \in \Delta : \mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq \{0\}\}$. *Suppose, in addition, that* $\mathfrak{e}$ *contains a subspace* $\mathfrak{h}_1 \subseteq \mathfrak{h}$ *separating the points of* $\operatorname{span}_{\mathbb{K}} \Delta_0$. *Then* $\mathfrak{g}_0$ *is a separated subalgebra of* $\mathfrak{g}$.

(b) *Suppose that* $\mathfrak{g}_0$ *is a separated subalgebra of* $\mathfrak{g}$. *Consider the root decomposition* $\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\beta\in\Delta(\mathfrak{g}_0,\mathfrak{h}_0)}(\mathfrak{g}_0)^\beta$ *of* $\mathfrak{g}_0$ *with respect to* $\mathfrak{h}_0$. *Then* $\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = r(\Delta_0)$ *where* $r\colon \operatorname{span}_{\mathbb{K}} \Delta_0 \to (\mathfrak{h}_0)^*$ *is the restriction map. If* $\alpha \in \Delta_i$ *and* $\pm\alpha \in \Delta_0$, *then* $r(\alpha) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$. *If, conversely,* $\alpha \in \Delta$ *is such that* $r(\alpha) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$, *then also* $\alpha \in \Delta_i$. *In both cases we have* $\mathfrak{g}^{\pm\alpha} = (\mathfrak{g}_0)^{\pm r(\alpha)}$ *and* $\check{\alpha} = r(\alpha)\check{}$. *In particular, the* $\mathfrak{h}_0$*-invariant Levi complement of* $\mathfrak{g}_0$ *is contained in* $\mathfrak{s} = \operatorname{span}_{\mathbb{K}} \check{\Delta}_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$.

Proof. (a) Since the endomorphisms $\operatorname{ad} h$ for $h \in \mathfrak{h}$ are derivations of $\mathfrak{g}$, the subalgebra $\mathfrak{g}_0$, which is generated by an $\mathfrak{h}$-invariant subspace of $\mathfrak{g}$, is itself $\mathfrak{h}$-invariant and thus adapted to the root decomposition of $\mathfrak{g}$. Moreover $\mathfrak{g}_0$ is finite dimensional because $\mathfrak{g}$ is locally finite.

If $\mathfrak{e}$ contains a subspace $\mathfrak{h}_1 \subseteq \mathfrak{h}$ separating the points of $\operatorname{span}_{\mathbb{K}} \Delta_0$, then the restriction map $r\colon \operatorname{span}_{\mathbb{K}} \Delta_0 \to (\mathfrak{h}_0)^*$ is injective. Therefore $\mathfrak{h}_0$ is maximal abelian and $(\mathfrak{g}_0)^{r(\alpha)} = \mathfrak{g}^\alpha \cap \mathfrak{g}_0$ for all $\alpha \in \Delta_0$ (Lemma III.3).

(b) Since $\mathfrak{h}_0$ is maximal abelian, we have $r(\alpha) \neq 0$ for all $\alpha \in \Delta_0$, so that $\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = r(\Delta_0)$ (Lemma III.3). If $\alpha \in \Delta_i$, then the root spaces $\mathfrak{g}^{\pm\alpha}$ are one dimensional, and $\pm\alpha \in \Delta_0$ implies that they are equal to $(\mathfrak{g}_0)^{r(\alpha)}$. Therefore we have $\pm r(\alpha) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$. If, on the other hand, $\alpha \in \Delta$ is such that $r(\alpha) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$, then $\mathfrak{g}^{\pm\alpha} \cap \mathfrak{g}_0 = (\mathfrak{g}_0)^{\pm r(\alpha)}$ implies that $\alpha \in \Delta_i$. Moreover, we have $\mathfrak{g}^{\pm\alpha} = (\mathfrak{g}_0)^{\pm r(\alpha)}$ in both cases because all root spaces are one dimensional, implying that $\check{\alpha} = r(\alpha)\check{}$. The last statement follows immediately. ∎

The proof of the existence of a generalized Levi decomposition of $\mathfrak{g}$ and the characterization of the locally finite split semisimple Lie algebras are somewhat interwoven. We first show that every perfect Lie algebra with only integrable roots is semisimple.

DEFINITION III.6. A subset Δ_0 of Δ is called *closed* if $(\Delta_0 + \Delta_0) \cap \Delta \subseteq \Delta_0$, and it is called *symmetric* if $\Delta_0 = -\Delta_0$. A closed and symmetric subset of Δ is called a *root subsystem*.

PROPOSITION III.7. *Suppose that* $\mathfrak{g}$ *has only integrable roots. Let* Δ_0 *be a finite root subsystem of* Δ *and set* $\mathfrak{g}_{\Delta_0} = \mathfrak{h}_{\Delta_0} + \sum_{\alpha\in\Delta_0} \mathfrak{g}^\alpha$ *where* $\mathfrak{h}_{\Delta_0} = \operatorname{span}_{\mathbb{K}} \check{\Delta}_0$. *Then* $\mathfrak{g}_{\Delta_0}$ *is a finite dimensional semisimple separated subalgebra of* $\mathfrak{g}$.

Proof. The subalgebra $\mathfrak{h}_{\Delta_0}$ of $\mathfrak{g}_{\Delta_0}$ is maximal abelian because no root of Δ_0 vanishes on $\mathfrak{h}_{\Delta_0}$, and hence $\mathfrak{h}_{\Delta_0}$ is a splitting Cartan subalgebra. If $\mathfrak{g}_{\Delta_0} = \mathfrak{h}_{\Delta_0} + \sum_{\beta\in\Delta(\mathfrak{g}_{\Delta_0},\mathfrak{h}_{\Delta_0})}(\mathfrak{g}_{\Delta_0})^\beta$ denotes the root decomposition of $\mathfrak{g}_{\Delta_0}$ with respect to $\mathfrak{h}_{\Delta_0}$, then for $\alpha \in \Delta_0$ we have $\mathfrak{g}^{\pm\alpha} \subseteq (\mathfrak{g}_{\Delta_0})^{\pm r(\alpha)}$ where

$r\colon \operatorname{span}_{\mathbb{K}} \Delta_0 \to (\mathfrak{h}_{\Delta_0})^*$ is the restriction map. From this we derive that all roots of $\Delta(\mathfrak{g}_{\Delta_0}, \mathfrak{h}_{\Delta_0}) = r(\Delta_0)$ are integrable. Thus we get $\mathfrak{g}^\alpha = (\mathfrak{g}_{\Delta_0})^{r(\alpha)}$ and $\check{\alpha} = r(\alpha)\check{}$ for $\alpha \in \Delta_0$ because both root spaces are one dimensional, showing that $\mathfrak{g}_{\Delta_0}$ is a finite dimensional separated subalgebra of $\mathfrak{g}$. Theorem II.1 implies that $\mathfrak{g}_{\Delta_0}$ is a semisimple Lie algebra. ∎

LEMMA III.8. *Let $\mathfrak{g}$ be a Lie algebra and E a subset of $\mathfrak{g}$. Then the subalgebra of $\mathfrak{g}$ generated by E is spanned by all left-normed products $[x_1, \ldots, x_n]$, which are recursively defined by $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$, where $x_1, \ldots, x_n \in E$.*

Proof (cf. [AmSt74, Chap. 1, Lemma 1.1]). The statement can be proved by induction. ∎

PROPOSITION III.9. *Suppose that $\mathfrak{g}$ has only integrable roots. Let $M \subseteq \Delta$ be a finite subset and $\mathfrak{g}_0$ the subalgebra of $\mathfrak{g}$ generated by the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in M$. Then $\mathfrak{g}_0$ is a finite dimensional semisimple separated subalgebra of $\mathfrak{g}$. Moreover, we have $\mathfrak{g}_0 = \mathfrak{g}_{\Delta_0}$ for a finite root subsystem Δ_0 of Δ, which satisfies* $\operatorname{span}_{\mathbb{K}} \Delta_0 = \operatorname{span}_{\mathbb{K}} M$ *and* $\operatorname{span}_{\mathbb{K}} \check{\Delta}_0 = \operatorname{span}_{\mathbb{K}} \check{M}$.

Proof. Since the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in M$ are finite dimensional, $\mathfrak{h}$-invariant, and perfect, the same holds for $\mathfrak{g}_0$ (Lemma III.5(a)). Therefore $\mathfrak{g}_0$ is adapted to the root decomposition of $\mathfrak{g}$, and, since $\dim \mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta$, equal to $\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha$ where $\Delta_0 = \{\alpha \in \Delta : \mathfrak{g}^\alpha \subseteq \mathfrak{g}_0\}$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$.

Since $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$, Proposition I.7(v) implies that Δ_0 is a closed subset of Δ. In order to see that Δ_0 is symmetric, we show that for $\alpha_1, \ldots, \alpha_n \in \Delta$ the relation $[\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_n}] \neq 0$ implies also that $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_n}] \neq 0$. We prove this statement by induction on n, the case $n = 1$ being clear. If $n > 1$ and $[\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_n}] \neq 0$, then setting $\beta := \sum_{j=1}^{n-1} \alpha_j$ we have $\{0\} \neq [\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_{n-1}}] \subseteq \mathfrak{g}^\beta$ and thus either $\beta \in \Delta$ or $\beta = 0$. In the first case we obtain $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_{n-1}}] = \mathfrak{g}^{-\beta}$ by the induction hypothesis and $-\beta, -\alpha_n \in \Delta$, $-(\beta + \alpha_n) \in \Delta \cup \{0\}$ by the symmetry of Δ. Hence, in view of Proposition I.7(v), we get that $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_n}] = [\mathfrak{g}^{-\beta}, \mathfrak{g}^{-\alpha_n}] \neq 0$. Otherwise, if $\beta = 0$, then we have $-\sum_{j=1}^{n-2} \alpha_j = \alpha_{n-1}$. Applying the induction hypothesis we obtain that $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_{n-1}}] = [\mathfrak{g}^{\alpha_{n-1}}, \mathfrak{g}^{-\alpha_{n-1}}] = \mathbb{K}\check{\alpha}_{n-1}$ and further that $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_n}] = \alpha_n(\check{\alpha}_{n-1})\mathfrak{g}^{-\alpha_n} \neq 0$. Now let $\beta \in \Delta_0$. Since $\mathfrak{g}_0$ is generated by the root spaces $\mathfrak{g}^{\pm\alpha}$ for $\alpha \in M$, we have $\mathfrak{g}^\beta = [\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_n}]$ where $\alpha_j \in \{\pm\alpha : \alpha \in M\}$ for $j = 1, \ldots, n$. From this $[\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_n}] \neq \{0\}$ follows, implying that $\mathfrak{g}^{-\beta} = [\mathfrak{g}^{-\alpha_1}, \ldots, \mathfrak{g}^{-\alpha_n}] \subseteq \mathfrak{g}_0$ and thus $-\beta \in \Delta_0$. Hence Δ_0 is a finite root subsystem of Δ.

The symmetry of Δ_0 and the perfectness of $\mathfrak{g}_0$ entail that $\mathfrak{h}_0 = \operatorname{span}_{\mathbb{K}} \check{\Delta}_0 = \mathfrak{h}_{\Delta_0}$, implying that $\mathfrak{g}_0$ is a finite dimensional semisimple separated subalgebra of $\mathfrak{g}$ (Proposition III.7).

We have $M \subseteq \Delta_0 \subseteq \operatorname{span}_{\mathbb{Z}} M$, showing that $\operatorname{span}_{\mathbb{K}} \Delta_0 = \operatorname{span}_{\mathbb{K}} M$. Moreover, the space $\operatorname{span}_{\mathbb{K}} \check{M} + \sum_{\alpha\in\Delta_0} \mathfrak{g}^\alpha$ contains the test algebras $\mathfrak{g}(\beta)$ for $\beta \in M$ and is invariant under them, implying that $\mathfrak{g}_0 \subseteq \operatorname{span}_{\mathbb{K}} \check{M} + \sum_{\alpha\in\Delta_0} \mathfrak{g}^\alpha$ and thus that $\mathfrak{h}_{\Delta_0} = \operatorname{span}_{\mathbb{K}} \check{\Delta}_0 = \operatorname{span}_{\mathbb{K}} \check{M}$. ∎

COROLLARY III.10. *For $\alpha, \beta \in \Delta$ we have $\alpha(\check{\beta}) \in \mathbb{Z}$ and $|\alpha(\check{\beta})| \leq 3$.*

Proof. For $\alpha, \beta \in \Delta$ let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by the test algebras $\mathfrak{g}(\alpha)$ and $\mathfrak{g}(\beta)$. Then $\mathfrak{g}_0$ is a finite dimensional semisimple separated subalgebra of $\mathfrak{g}$ (Proposition III.9). Using the results and the terminology of Lemma III.5, we get that $r(\alpha)(r(\beta)\check{}) \in \mathbb{Z}$ and $|r(\alpha)(r(\beta)\check{})| \leq 3$ because these relations hold in the finite root system $\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = r(\Delta_0)$ of semisimple type. Now the statement follows from $\alpha(\check{\beta}) = r(\alpha)(r(\beta)\check{})$. ∎

THEOREM III.11. *Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite split Lie algebra with root system Δ. If all roots of Δ are integrable and $\mathfrak{h} = \operatorname{span}_{\mathbb{K}} \check{\Delta}$, then $\mathfrak{g}$ is semisimple in the sense that it is a direct sum of simple ideals.*

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. We claim that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, as the centralizer of an ideal, is an ideal of $\mathfrak{g}$. The invariance of $\mathfrak{a}$ under $\mathfrak{h}$ and $\dim \mathfrak{g}^\alpha = 1$ for each root α imply that $\mathfrak{a} = (\mathfrak{h} \cap \mathfrak{a}) \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{a}}} \mathfrak{g}^\alpha$, where $\Delta_{\mathfrak{a}} = \{\alpha \in \Delta\colon \mathfrak{g}^\alpha \subseteq \mathfrak{g}\}$. For $\alpha \in \Delta_{\mathfrak{a}}$ we have $\mathbb{K}\check{\alpha} = [\mathfrak{g}^{-\alpha}, \mathfrak{g}^\alpha] \subseteq [\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$ and further $\mathfrak{g}^{-\alpha} = [\mathfrak{g}^{-\alpha}, \mathbb{K}\check{\alpha}] \subseteq \mathfrak{a}$, implying that $\Delta_{\mathfrak{a}}$ is symmetric and that $\operatorname{span}_{\mathbb{K}} \check{\Delta}_{\mathfrak{a}} \subseteq \mathfrak{h} \cap \mathfrak{a}$. Let $\alpha \in \Delta_{\mathfrak{a}}$ and $\beta \in \Delta_{\mathfrak{b}} := \Delta \backslash \Delta_{\mathfrak{a}}$. Then the subalgebra $\mathfrak{g}_0$ generated by $\mathfrak{g}(\alpha)$ and $\mathfrak{g}(\beta)$ is finite dimensional and semisimple (Proposition III.9). Therefore we have $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{a}) \oplus \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{g}_0 \cap \mathfrak{a})$. From $\beta \notin \Delta_{\mathfrak{a}}$ it follows that $\mathfrak{g}(\beta)$ intersects $\mathfrak{a}$ trivially and thus $\mathfrak{g}(\beta) \subseteq \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{g}_0 \cap \mathfrak{a})$, so that $[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] = \{0\}$. Hence $\mathfrak{b} := \operatorname{span}_{\mathbb{K}} \check{\Delta}_{\mathfrak{b}} + \sum_{\alpha\in\Delta_{\mathfrak{b}}} \mathfrak{g}^\alpha$ is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, and therefore $\mathfrak{g} = \mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$. Moreover $\mathfrak{a} \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathfrak{z}(\mathfrak{g})$ follows. Suppose $x \in \mathfrak{z}(\mathfrak{g}) \subseteq \operatorname{span}_{\mathbb{K}} \check{\Delta}$. Then there exists a finite subset $M \subseteq \Delta$ such that $x \in \operatorname{span}_{\mathbb{K}} \check{M}$. Since the Lie algebra $\mathfrak{g}_0 := \langle\langle \mathfrak{g}(\alpha) : \alpha \in M \rangle\rangle$ is semisimple, we see that $x \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_0 \subseteq \mathfrak{z}(\mathfrak{g}_0) = \{0\}$. This proves that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, showing that any ideal of $\mathfrak{g}$ has a complementary ideal and thus that $\mathfrak{g}$ is semisimple (cf. [La74, Chap. XVII, Sect. 2]). ∎

THEOREM III.12. *Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite split Lie algebra with root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$. The subspace $\mathfrak{s} = \operatorname{span}_{\mathbb{K}} \check{\Delta}_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$ is a semisimple subalgebra of $\mathfrak{g}$. In particular, $\mathfrak{z}(\mathfrak{s}) = \{0\}$.*

Proof. In order to show that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$, it suffices to see that for $\gamma, \delta \in \Delta_i$ we have $[\mathfrak{g}^\gamma, \mathfrak{g}^\delta] \subseteq \mathfrak{s}$. For this, let $\mathfrak{h}_1$ be a finite dimensional subspace of $\mathfrak{h}$ separating the points of $\operatorname{span}_{\mathbb{K}}\{\gamma, \delta\}$, and let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}_1$, $\mathfrak{g}(\gamma)$, and $\mathfrak{g}(\delta)$. Then Lemma III.5 implies that $\mathfrak{g}_0$ is a separated subalgebra of $\mathfrak{g}$. Using its terminology, we derive that $r(\gamma), r(\delta) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$ and, furthermore, that $[\mathfrak{g}^\gamma, \mathfrak{g}^\delta] =$

$[(\mathfrak{g}_0)^{r(\gamma)}, (\mathfrak{g}_0)^{r(\delta)}] \subseteq [\mathfrak{s}_0, \mathfrak{s}_0] \subseteq \mathfrak{s}_0 \subseteq \mathfrak{s}$. The subalgebra $\mathfrak{s}$ is semisimple according to Theorem III.11. ∎

We will see in Theorem III.19 that also the converse of Theorem III.11 holds. The proof of Theorem III.19 requires the existence of a generalized Levi decomposition for a locally finite split Lie algebra, which we aim at next.

DEFINITION III.13. A Lie algebra is called *locally solvable*, respectively *locally nilpotent*, if every finite subset of it is contained in a solvable, respectively nilpotent, subalgebra.

THEOREM III.14. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a locally finite split Lie algebra with root decomposition* $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$.

(a) *The space* $\mathfrak{n} := \mathfrak{z}(\mathfrak{g}) + \sum_{\alpha \in \Delta_n} \mathfrak{g}^\alpha$ *is the unique maximal locally nilpotent ideal of* $\mathfrak{g}$.

(b) *The space* $\mathfrak{r} := \mathfrak{z}_{\mathfrak{h}}(\mathfrak{s}) + \sum_{\alpha \in \Delta_n} \mathfrak{g}^\alpha$ *is the unique maximal locally solvable ideal of* $\mathfrak{g}$.

Proof. (a) First we show that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$. Since $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$ by definition, we have to show that for $\alpha \in \Delta$ and $\beta \in \Delta_n$ we have $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{n}$. Suppose that $\alpha = -\beta$. Then for non-zero root vectors $X_{\pm\beta} \in \mathfrak{g}^{\pm\beta}$ the test algebra $\mathfrak{g}(X_\beta, X_{-\beta})$ is solvable. According to Lie's Theorem, the elements of its commutator algebra act nilpotently on every finite dimensional $\mathfrak{g}(X_\beta, X_{-\beta})$-module. Since $\mathfrak{g}$ is a locally finite $\mathfrak{g}(X_\beta, X_{-\beta})$-module, this implies that the endomorphism $\operatorname{ad}[X_\beta, X_{-\beta}]$ is locally nilpotent. At the same time $\operatorname{ad}[X_\beta, X_{-\beta}]$ is diagonalizable, hence zero, implying that $[X_\beta, X_{-\beta}] \in \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{n}$. If $\alpha \neq -\beta$, it suffices to verify that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq \{0\}$ entails that $\alpha + \beta$ is not an integrable root. Suppose, on the contrary, that $\alpha + \beta \in \Delta_i$. Let $X_\alpha \in \mathfrak{g}^\alpha$ and $X_\beta \in \mathfrak{g}^\beta$ with $[X_\alpha, X_\beta] \neq 0$ and choose a finite dimensional subspace $\mathfrak{h}_1 \subseteq \mathfrak{h}$ separating the elements in $\operatorname{span}_{\mathbb{K}}\{\alpha, \beta\}$. Consider the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ generated by $\mathfrak{h}_1$, $\mathfrak{g}(\alpha + \beta)$, X_α, and X_β. Then $\mathfrak{g}_0$ is a finite dimensional separated subalgebra of $\mathfrak{g}$ (Lemma III.5(a)). Therefore we get $r(\alpha) + r(\beta) = r(\alpha + \beta) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$ (Lemma III.5(b)), so that Theorem II.1 implies that $r(\beta) \in \Delta(\mathfrak{g}_0, \mathfrak{h}_0)_i$. We derive that $\beta \in \Delta_i$, which contradicts $\beta \in \Delta_n$. This proves that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$.

To see that $\mathfrak{n}$ is locally nilpotent, it suffices to show that every finite dimensional $\mathfrak{h}$-invariant subalgebra $\mathfrak{m} \subseteq \mathfrak{n}$ is nilpotent. Let $\mathfrak{m} \subseteq \mathfrak{n}$ be such a subalgebra. Then $\mathfrak{m} = \big(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}\big) + \sum_{\alpha \in \Delta_n} (\mathfrak{g}^\alpha \cap \mathfrak{m})$ and the set $\Delta_{\mathfrak{m}} := \{\alpha \in \Delta_n\colon \mathfrak{g}^\alpha \cap \mathfrak{m} \neq \{0\}\}$ of all roots contributing to $\mathfrak{m}$ is finite, so that we find a finite dimensional subspace $\mathfrak{h}_{\mathfrak{m}} \subseteq \mathfrak{h}$ separating the points in $\operatorname{span}_{\mathbb{K}} \Delta_{\mathfrak{m}}$. Then $\mathfrak{g}_0 := \mathfrak{h}_{\mathfrak{m}} + \mathfrak{m}$ is a finite dimensional split Lie algebra with the splitting Cartan subalgebra $\mathfrak{h}_0 := \mathfrak{h}_{\mathfrak{m}} + \big(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}\big)$ that separates the points of $\operatorname{span}_{\mathbb{K}} \Delta_{\mathfrak{m}}$. No roots of $\Delta(\mathfrak{g}_0, \mathfrak{h}_0)$ are integrable by Lemma III.5(b) and $\mathfrak{h}_0 \cap \mathfrak{m} = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m} \subseteq \mathfrak{z}(\mathfrak{g}_0)$, showing that $\mathfrak{m}$ is contained in the nilradical $\mathfrak{n}_0$ of $\mathfrak{g}_0$ (Theorem II.1). Therefore $\mathfrak{m}$ is nilpotent, implying that the ideal $\mathfrak{n}$ is locally nilpotent.

Let $\mathfrak{n}' \subseteq \mathfrak{g}$ be a locally nilpotent ideal. Then $\mathfrak{n}'$ intersects every simple test algebra $\mathfrak{g}(\alpha)$, $\alpha \in \Delta_i$ trivially and therefore is contained in $\mathfrak{h} + \mathfrak{n}$. If $\mathfrak{h} \cap \mathfrak{n}' \not\subseteq \mathfrak{z}(\mathfrak{g}) = \bigcap_{\alpha\in\Delta} \ker\alpha$, then there exists an element $H \in \mathfrak{h} \cap \mathfrak{n}'$ and a root $\alpha \in \Delta$ with $\alpha(H) \neq 0$. But then for each $x_\alpha \in \mathfrak{g}^\alpha$ the space $\mathbb{K}H + x_\alpha$ is a solvable subalgebra of $\mathfrak{n}'$ which is not nilpotent, contradicting the local nilpotency of $\mathfrak{n}'$. Hence $\mathfrak{n}' \subseteq \mathfrak{n}$, showing that $\mathfrak{n}$ is the unique maximal locally nilpotent ideal of $\mathfrak{g}$.

(b) Writing $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \mathfrak{s}$ and $\mathfrak{r} = \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) + \mathfrak{n}$, we see that $\mathfrak{r}$ is an ideal of $\mathfrak{g}$ because $[\mathfrak{g}, \mathfrak{r}] \subseteq [\mathfrak{h} + \mathfrak{n} + \mathfrak{s}, \mathfrak{z}_\mathfrak{h}(\mathfrak{s})] + [\mathfrak{g}, \mathfrak{n}] \subseteq \mathfrak{n} \subseteq \mathfrak{r}$. Further we see that $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{n}$, showing that $[\mathfrak{r}, \mathfrak{r}]$ is locally nilpotent. Hence the commutator algebra $\mathbb{D}\,\mathfrak{f}$ of each finite dimensional subalgebra of $\mathfrak{f}$ of $\mathfrak{r}$ is nilpotent, $\mathfrak{f}$ is therefore solvable and hence $\mathfrak{r}$ is locally solvable.

If $\mathfrak{r}' \subseteq \mathfrak{g}$ is a locally solvable ideal, then $\mathfrak{r}'$ intersects all subalgebras $\mathfrak{g}(\alpha)$, $\alpha \in \Delta_i$, trivially, so that it is contained in $\mathfrak{h} + \mathfrak{n}$. Further the fact that $\mathfrak{r}'$ is an ideal not containing any root space of an integrable root implies that each integrable root vanishes on $\mathfrak{h} \cap \mathfrak{r}'$, showing that $\mathfrak{r}' \subseteq \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) + \mathfrak{n} = \mathfrak{r}$. Thus $\mathfrak{r}$ is the unique maximal locally solvable ideal of $\mathfrak{g}$. ∎

Remark III.15 [AmSt74, Chap. 6, Theorem 1.3]. states that any Lie algebra contains a unique maximal locally nilpotent ideal, which is called the *Hirsch–Plotkin radical*, and that a locally finite Lie algebra contains a unique maximal locally solvable ideal.

THEOREM III.16 (Generalized Levi decomposition). *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a locally finite split Lie algebra with root decomposition* $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$. *For a vector space complement* $\mathfrak{a}$ *of* $\mathfrak{z}_\mathfrak{h}(\mathfrak{s}) + \operatorname{span}_\mathbb{K} \check{\Delta}_i$ *in* $\mathfrak{h}$, *we have* $\mathfrak{g} \cong \mathfrak{r} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$. *If* $\mathfrak{s}$ *is finite dimensional, then* $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$ *and, in particular,* $\mathfrak{h} = \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) \oplus \operatorname{span}_\mathbb{K} \check{\Delta}_i$.

Proof. First we observe that $\mathfrak{z}_\mathfrak{h}(\mathfrak{s}) \cap \operatorname{span}_\mathbb{K} \check{\Delta}_i \subseteq \mathfrak{z}(\mathfrak{s}) = \{0\}$ by Theorem III.12. Hence $\mathfrak{h} = \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) \oplus \operatorname{span}_\mathbb{K} \check{\Delta}_i \oplus \mathfrak{a}$ and $\mathfrak{g}$ decomposes as the direct vector space sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s} \oplus \mathfrak{a}$. Therefore $[\mathfrak{a}, \mathfrak{s}] \subseteq [\mathfrak{h}, \mathfrak{s}] \subseteq \mathfrak{s}$ implies that $\mathfrak{s} \rtimes \mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{g} \cong \mathfrak{r} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$ follows immediately.

If, in addition, $\mathfrak{s}$ is finite dimensional, then all derivations of $\mathfrak{s}$ are inner and $\mathfrak{z}(\mathfrak{s}) = \{0\}$, implying that the homomorphism $\operatorname{ad}_\mathfrak{s} : \mathfrak{h} + \mathfrak{s} \to \operatorname{der} \mathfrak{s}$ maps $\mathfrak{s}$ bijectively onto $\operatorname{der} \mathfrak{s}$. Therefore $\mathfrak{h} + \mathfrak{s} \cong (\ker \operatorname{ad}_\mathfrak{s} \cap (\mathfrak{h} + \mathfrak{s})) \oplus \operatorname{der} \mathfrak{s} \cong \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) \oplus \mathfrak{s} = (\mathfrak{h} \cap \mathfrak{r}) \oplus \mathfrak{s}$. Thus $\mathfrak{h} \subseteq \mathfrak{r} + \mathfrak{s}$ and therefore $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ and $\mathfrak{h} = \mathfrak{z}_\mathfrak{h}(\mathfrak{s}) \oplus \operatorname{span}_\mathfrak{f} \check{\Delta}_i$. ∎

DEFINITION III.17. The semisimple subalgebra $\mathfrak{s} = \operatorname{span}_\mathbb{K} \check{\Delta}_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$ is called the $\mathfrak{h}$-*invariant generalized Levi subalgebra* of the split Lie algebra $\mathfrak{g}$.

Remark III.18. If the $\mathfrak{h}$-invariant generalized Levi subalgebra $\mathfrak{s}$ of a locally finite split Lie algebra $\mathfrak{g}$ is infinite dimensional, then we cannot expect that $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$, as the example $\mathfrak{g} = \mathfrak{gl}(\mathbb{N}, \mathbb{K})$ shows (cf. Example I.4(c)). Here we have $\mathfrak{z}(\mathfrak{g}) = \{0\}$ because a matrix of $\mathfrak{g}$ has only finitely many non-zero entries, $\mathfrak{r} = \{0\}$ and $\mathfrak{s} = \mathfrak{sl}(\mathbb{N}, \mathbb{K})$, showing, in particular, that there is no obvious analog of Weyl's Theorem for the Lie algebra $\mathfrak{sl}(\mathbb{N}, \mathbb{K})$. We cannot copy the proof for the case where $\dim \mathfrak{s} < \infty$ given in Theorem III.16 because, in general, $\operatorname{der} \mathfrak{s}$ is much bigger than $\operatorname{ad} \mathfrak{s}$. For example, each function $f \colon \mathbb{N} \to \mathbb{K}$ yields a derivation of $\mathfrak{sl}(\mathbb{N}, \mathbb{K})$ by $D_f(E_{ij}) = (f(i) - f(j))E_{ij}$ for $i, j \in \mathbb{N}$ that is, in general, not of the form $\operatorname{ad} x$ for an element $x \in \mathfrak{sl}(\mathbb{N}, \mathbb{K})$. Even if $\mathfrak{s} = \oplus_{j \in J} \mathfrak{s}_j$ where J is an infinite set and $\mathfrak{s}_j$ is a finite dimensional simple ideal of $\mathfrak{s}$ for $j \in J$, we have

$$\operatorname{ad} \mathfrak{s} = \oplus_{j \in J} \operatorname{der} \mathfrak{s}_j \subseteq \Pi_{j \in J} \operatorname{der} \mathfrak{s}_j = \operatorname{der} \mathfrak{s}.$$

Here the last equation holds because for a derivation $D \in \operatorname{der} \mathfrak{s}$, a simple ideal $\mathfrak{s}_j$, $j \in J$, and $x, y \in \mathfrak{s}_j$, we have $D([x, y]) = [D(x), y] + [x, D(y)] \subseteq [\mathfrak{s}, \mathfrak{s}_j] \subseteq \mathfrak{s}_j$, showing that $D(\mathfrak{s}_j) \subseteq \mathfrak{s}_j$ and thus that $D = (D|_{\mathfrak{s}_j})_{j \in J} \in \Pi_{j \in J} \operatorname{der} \mathfrak{s}_j$. With $\mathfrak{h}_1 := \Pi_{j \in J} \operatorname{ad}(\mathfrak{h} \cap \mathfrak{s}_j)$ we obtain a split Lie algebra $\mathfrak{s}_1 := \mathfrak{s} \rtimes \mathfrak{h}_1$ with a split subalgebra $\mathfrak{h} \oplus \mathfrak{h}_1$, $\mathfrak{r} = \{(h, -\operatorname{ad} h) : h \in \mathfrak{h}\}$, and $\mathfrak{r} + \mathfrak{s} = \mathfrak{s} \rtimes \operatorname{ad} \mathfrak{h} \neq \mathfrak{s}_1$.

The existence of a Levi decomposition in another class of locally finite Lie algebras has been investigated in [AmSt74, Chap. 13.5]. There the Lie algebras are assumed to be neoclassical, which means that they are generated by a set of finite dimensional local subideals, but they are not assumed to have a root decomposition. It is shown that if $\mathfrak{g}$ is a neoclassical Lie algebra such that the maximal locally solvable ideal $\mathfrak{r}$, respectively the space $\mathfrak{r} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r})$, has finite codimension, then $\mathfrak{g}$ has a Levi decomposition. Moreover, it follows that if $\mathfrak{g}$ is a locally finite Lie algebra such that $\mathfrak{r}$ is finite dimensional and $\mathfrak{g}/\mathfrak{r}$ is the direct sum of finite dimensional simple ideals, then $\mathfrak{g}$ has also a Levi decomposition. It should be mentioned at this point that it is, in general, not clear which locally finite split Lie algebras are neoclassical.

THEOREM III.19. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a locally finite split Lie algebra with root decomposition* $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$. *Then the following are equivalent*:

(1) *The Lie algebra* $\mathfrak{g}$ *is semisimple*.

(2) *All roots are integrable and* $\mathfrak{h} = \operatorname{span}_{\mathbb{K}} \check{\Delta}$.

(3) *The Lie algebra* $\mathfrak{g}$ *is perfect and all roots are integrable*.

Proof. (1) $\Rightarrow$ (2) We may assume w.l.o.g. that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ has nilpotent roots, then Theorem III.14 implies that $\mathfrak{g} = \mathfrak{n}$ and thus that $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ is a splitting Cartan subalgebra. But the simplicity of $\mathfrak{g}$ entails that $\mathfrak{z}(\mathfrak{g}) = \{0\}$, which is absurd. Therefore all roots of Δ are integrable.

Moreover $\mathrm{span}_{\mathbb{K}} \check{\Delta} = \sum_{\alpha\in\Delta}[\mathfrak{g}^{-\alpha}, \mathfrak{g}^{\alpha}] = \mathbb{D}\mathfrak{g} \cap \mathfrak{h} = \mathfrak{h}$, where the last equality holds because $\mathfrak{g}$ is simple.

The implication (2)$\Rightarrow$(1) follows from Theorem III.16.

The equivalence of (2) and (3) is immediate. ∎

COROLLARY III.20. *The maximal locally nilpotent ideal* $\mathfrak{n}$ *of* $\mathfrak{g}$ *is not simple*.

Proof. The statement follows from the proof of Theorem III.19. For a more general argument we refer to [BaSt95, Corollary 3.2], where it is shown that a simple locally finite Lie algebra is not locally solvable, hence in particular not locally nilpotent. ∎

DEFINITION III.21. A subset $M \subseteq \Delta$ is called *irreducible* if for every two roots $\alpha, \beta \in M$ there exists a chain of roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that $\alpha_j(\check{\alpha}_{j+1}) \neq 0$ for $j = 1, \ldots, n-1$.

PROPOSITION III.22. *Suppose that* $\mathfrak{g}$ *is semisimple*. *Then* $\mathfrak{g}$ *is simple if and only if the root system* Δ *is irreducible*.

Proof. The statement can be proved as in the finite dimensional case. ∎

Remark III.23. In the forthcoming paper [NeSt99] we will classify the locally finite split simple Lie algebras. It will be shown that each infinite dimensional locally finite split simple Lie algebra is isomorphic to one of the Lie algebras $\mathfrak{sl}(J, \mathbb{K})$, $\mathfrak{o}(J, \mathbb{K})$, or $\mathfrak{sp}(J, \mathbb{K})$ where J is an infinite set whose cardinality equals the dimension of $\mathfrak{g}$.

IV. SPLIT GRADED LIE ALGEBRAS

In this section the more general framework of split graded Lie algebras is introduced, which links this paper to the setting of [Ne98]. We will see that every split graded Lie algebra can be extended by a space of derivations to a split Lie algebra (while a split Lie algebra is always split graded). Using the results of Section III, we formulate a structure theorem for locally finite split graded Lie algebras and give a characterization of locally finite split graded semisimple Lie algebras. Both can be done without much additional effort.

DEFINITION IV.1. Let Q be a torsion free abelian group and $\mathfrak{g} = \sum_{\alpha\in Q} \mathfrak{g}^{\alpha}$ a Q-graded Lie algebra. We call $\mathfrak{g}$ *split graded* if the subalgebra $\mathfrak{g}^0$ is abelian and there exists a map $\iota\colon Q \to (\mathfrak{g}^0)^*$ such that

$$[h, x_\alpha] = \iota(\alpha)(h)x_\alpha \qquad \text{for} \qquad h \in \mathfrak{g}^0, x_\alpha \in \mathfrak{g}^{\alpha}.$$

This means in particular that the endomorphisms $\operatorname{ad} h$ for $h \in \mathfrak{g}^0$ are simultaneously diagonalizable by the gradation of $\mathfrak{g}$, but the $\mathfrak{g}^0$-weight spaces might be strictly larger than the spaces $\mathfrak{g}^\alpha$, $\alpha \in Q$. The elements of

$$\Delta := \{\alpha \in Q \setminus \{0\} \colon \mathfrak{g}^\alpha \neq \{0\}\}$$

are called the *roots* of $\mathfrak{g}$. In the following we simplify our notation by writing $\alpha(h) := \iota(\alpha)(h)$.

Remark IV.2. The condition that the group Q is torsion free means that the natural map $Q \to \mathbb{Q} \otimes_{\mathbb{Z}} Q$ is injective, so that furthermore the map $Q \to \mathbb{K} \otimes_{\mathbb{Z}} Q$ is injective because $\mathbb{K} \otimes_{\mathbb{Z}} Q \cong \mathbb{K} \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes_{\mathbb{Z}} Q)$. The property that Q can be embedded into a $\mathbb{K}$-vector space implies that the space $\operatorname{Hom}(Q, \mathbb{K})$ of homomorphisms of Q into the additive group $(\mathbb{K}, +)$ separates the points of Q. As we will see in the next remark, this allows extension of a split graded Lie algebra by a subspace of $\operatorname{Hom}(Q, \mathbb{K})$ such that the extended Lie algebra contains a splitting Cartan subalgebra.

Remark IV.3. (a) Let $\mathfrak{g}$ be a split Lie algebra with a root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$. Then we obtain a gradation of $\mathfrak{g}$ by $Q := \operatorname{span}_{\mathbb{Z}} \Delta$, the subgroup of $\mathfrak{h}^*$ generated by Δ, turning $\mathfrak{g}$ into a split graded Lie algebra with $\mathfrak{g}^0 = \mathfrak{h}$.

(b) Suppose, conversely, that $\mathfrak{g} = \mathfrak{g}^0 + \sum_{\alpha\in Q} \mathfrak{g}^\alpha$ is a split graded Lie algebra. If the map $\iota\colon Q \to (\mathfrak{g}^0)^*$ is injective, then $\mathfrak{g}^0$ is maximal abelian and thus a splitting Cartan subalgebra of $\mathfrak{g}$. Moreover the gradation of $\mathfrak{g}$ coincides with the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{g}^0$. Otherwise we can extend $\mathfrak{g}$ to a split Lie algebra in the following way. Assign to each element $f \in \operatorname{Hom}(Q, \mathbb{K})$ the derivation $D_f \in \operatorname{der}(\mathfrak{g})$ given by $D_f \cdot x_\alpha = f(\alpha)x_\alpha$ for $x_\alpha \in \mathfrak{g}^\alpha$. If $\mathfrak{d} \subseteq \operatorname{Hom}(Q, \mathbb{K})$ is a subspace separating the points of Q, whose existence is guaranteed by Remark IV.2, then the Lie algebra $\mathfrak{g}_{\mathfrak{d}} := \mathfrak{g} \rtimes \mathfrak{d}$, endowed with the additional brackets $[f, x_\alpha] = D_f(x_\alpha)$ for $f \in \mathfrak{d}$ and $x_\alpha \in \mathfrak{g}^\alpha$, is a split Lie algebra with splitting Cartan subalgebra $\mathfrak{h} := \mathfrak{g}^0 \oplus \mathfrak{d}$. The root spaces of $\mathfrak{g}_{\mathfrak{d}}$ with respect to $\mathfrak{h}$ correspond to the homogeneous spaces of $\mathfrak{g}$. To use a more precise notation, this means that for $\alpha \in \Delta$ we have a linear functional, also denoted by α, which is given through $\alpha|_{\mathfrak{g}^0} = \iota(\alpha)$ and $\alpha(f) = f(\alpha)$ for $f \in \mathfrak{d}$ and satisfies $(\mathfrak{g}_{\mathfrak{d}})^\alpha = \mathfrak{g}^\alpha$.

Remark IV.4. An advantage of split graded Lie algebras in comparison to split Lie algebras is that a graded subalgebra of a split graded Lie algebra $\mathfrak{g}$ is split graded with respect to the gradation inherited from $\mathfrak{g}$. In contrast, an $\mathfrak{h}$-invariant subalgebra of a split Lie algebra $\mathfrak{g}$ with splitting Cartan subalgebra $\mathfrak{h}$ need not have a splitting Cartan subalgebra. (This was the reason for introducing separated subalgebras in Definition III.4.)

EXAMPLE IV.5. The Heisenberg algebra $\mathfrak{g} = \operatorname{span}_{\mathbb{K}}\{l(X_n), \partial/\partial X_n, \mathbf{1} : n \in \mathbb{N}\}$ that was introduced in Example I.4(a), is a split graded but not a split Lie algebra. The extension described in Example I.4(a) is a split Lie algebra.

The notion of test algebras and root types remain the same in the context of split graded Lie algebras; Lemma I.2 and Definition I.3 can be transferred literally (cf. [Ne98, Section I]).

LEMMA IV.6. *Let $\mathfrak{g} = \mathfrak{g}^0 + \sum_{\alpha\in Q} \mathfrak{g}^\alpha$ be a locally finite split graded Lie algebra with only integrable roots. Then $\mathfrak{g}^0$ is a splitting Cartan subalgebra and $\mathfrak{g}$ thus a split Lie algebra. Moreover the map $\iota\colon Q \to (\mathfrak{g}^0)^*$ is injective, so that the $\mathfrak{g}^0$-root decomposition coincides with the gradation.*

Proof. First we note that $(\mathbb{D}\,\mathfrak{g})^0 = \sum_{\alpha\in\Delta}[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \operatorname{span}_{\mathbb{K}} \check{\Delta} \subseteq \mathfrak{g}^0$ follows from $\Delta = \Delta_i$. This implies, in particular, that $\mathfrak{g}^0$ is maximal abelian and therefore a splitting Cartan subalgebra of $\mathfrak{g}$.

Let $\gamma, \delta \in \Delta$. We will show that $\iota(\gamma) \neq \iota(\delta)$, so that the map ι is injective and the statements follow. For this, extend $\mathfrak{g}$ to a split Lie algebra $\mathfrak{g}_{\mathfrak{d}} = \mathfrak{g} \rtimes \mathfrak{d}$, where $\mathfrak{d} \subseteq \operatorname{Hom}(Q, \mathbb{K})$ is a subspace separating the points of Δ. Consider the splitting Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^0 \oplus \mathfrak{d}$ of $\mathfrak{g}_{\mathfrak{d}}$ and the roots $\gamma, \delta \in \Delta(\mathfrak{g}_{\mathfrak{d}}, \mathfrak{h})$ corresponding to γ, δ (cf. Remark IV.3(b)). By Proposition III.9 the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_{\mathfrak{d}}$ that is generated by the test algebras $\mathfrak{g}_{\mathfrak{d}}(\gamma)$ and $\mathfrak{g}_{\mathfrak{d}}(\delta)$ is finite dimensional semisimple and contains $\mathfrak{h}_0 = \mathfrak{g}_{\mathfrak{d}} \cap \mathfrak{g}_0$ as a splitting Cartan subalgebra. Moreover $\mathfrak{g}^\gamma = (\mathfrak{g}_{\mathfrak{d}})^\gamma = (\mathfrak{g}_0)^{\gamma|_{\mathfrak{h}_0}}$ and $\mathfrak{g}^\delta = (\mathfrak{g}_0)^{\delta|_{\mathfrak{h}_0}}$ are different root spaces, showing that $\iota(\gamma)|_{\mathfrak{h}_0} = \gamma|_{\mathfrak{h}_0} \neq \delta|_{\mathfrak{h}_0} = \iota(\delta)|_{\mathfrak{h}_0}$. ∎

THEOREM IV.7 (Structure theorem for locally finite split graded Lie algebras). *Let $\mathfrak{g} = \mathfrak{g}^0 + \sum_{\alpha\in Q} \mathfrak{g}^\alpha$ be a locally finite split graded Lie algebra and $\mathfrak{s} = \operatorname{span}_{\mathbb{K}} \check{\Delta}_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$.*

(a) *The subspace $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$.*

(b) *The space $\mathfrak{n} := \mathfrak{z}(\mathfrak{g}) + \sum_{\alpha\in\Delta_n} \mathfrak{g}^\alpha$ is the unique maximal locally nilpotent ideal of $\mathfrak{g}$.*

(c) *The space $\mathfrak{r} := \mathfrak{z}_{\mathfrak{g}^0}(\mathfrak{s}) + \sum_{\alpha\in\Delta_n} \mathfrak{g}^\alpha$ is the unique maximal locally solvable ideal of $\mathfrak{g}$.*

(d) (Generalized Levi decomposition) *If $\mathfrak{a}$ is a vector space complement of $\mathfrak{z}_{\mathfrak{g}^0}(\mathfrak{s}) + \operatorname{span}_{\mathbb{K}} \check{\Delta}_i$ in $\mathfrak{g}^0$, then $\mathfrak{g} \cong \mathfrak{r} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$.*

(e) *If $\mathfrak{s}$ is finite dimensional, then $\mathfrak{s}$ is the unique graded Levi complement in $\mathfrak{g}$. In particular, we have $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$.*

Proof. Extend $\mathfrak{g}$ by a subspace $\mathfrak{d} \subseteq \operatorname{Hom}(Q, \mathbb{K})$ such that $\mathfrak{g}_{\mathfrak{d}} = \mathfrak{g} \rtimes \mathfrak{d}$ is a split Lie algebra with splitting Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^0 \oplus \mathfrak{d}$.

(a) The subspace $\mathfrak{s}$ is equal to the $\mathfrak{h}$-invariant generalized Levi complement of $\mathfrak{g}_{\mathfrak{d}}$ and hence a subalgebra of $\mathfrak{g}$.

(b) It is easy to check that $\mathfrak{n} = \mathfrak{n}_{\mathfrak{d}} \cap \mathfrak{g}$, where $\mathfrak{n}_{\mathfrak{d}}$ is the maximal locally nilpotent ideal of $\mathfrak{g}_{\mathfrak{d}}$, implying that $\mathfrak{n}$ is a locally nilpotent ideal of $\mathfrak{g}$. Consider the quotient algebra $\mathfrak{g}/\mathfrak{n} \cong (\mathfrak{g}^0 + \mathfrak{s})/\mathfrak{z}(\mathfrak{g})$. According to Lemma IV.6, the subalgebra $\mathfrak{g}^0 + \mathfrak{s}$, only having integrable roots, is a split Lie algebra, so that Theorem III.14(a) implies that the maximal locally nilpotent ideal of $(\mathfrak{g}^0 + \mathfrak{s})/\mathfrak{z}(\mathfrak{g})$ is trivial. This entails that every locally nilpotent ideal of $\mathfrak{g}$ is contained in $\mathfrak{n}$, saying that $\mathfrak{n}$ is the unique maximal locally nilpotent ideal of $\mathfrak{g}$.

(c) The proof is similar to that of (b).

(d) Using Theorem III.12, we get

$$\mathfrak{z}_{\mathfrak{g}^0}(\mathfrak{s}) \cap \operatorname{span}_{\mathbb{K}} \check{\Delta}_i \subseteq \mathfrak{z}_{\mathfrak{h}}(\mathfrak{s}) \cap \operatorname{span}_{\mathbb{K}} \check{\Delta}_i = \{0\},$$

implying that we have a direct sum of vector spaces $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s} \oplus \mathfrak{a}$. From this the statement follows immediately.

(e) If $\mathfrak{s}$ is finite dimensional, then $\mathfrak{g}_{\mathfrak{d}} = \mathfrak{r}_{\mathfrak{d}} \rtimes \mathfrak{s}$ by Theorem III.16. This implies that $\mathfrak{g} = (\mathfrak{r}_{\mathfrak{d}} \cap \mathfrak{g}) \rtimes \mathfrak{s} = \mathfrak{r} \rtimes \mathfrak{s}$. Hence $\mathfrak{s}$ is a graded Levi complement of $\mathfrak{g}$. Suppose $\mathfrak{s}'$ is another graded Levi complement of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}'$ implies that $\sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha \subseteq \mathfrak{s}'$, further $\mathfrak{s} \subseteq \mathfrak{s}'$, and therefore $\mathfrak{s} = \mathfrak{s}'$. ∎

LEMMA IV.8. *Let* $\mathfrak{g} = \mathfrak{g}^0 + \sum_{\alpha\in Q} \mathfrak{g}^\alpha$ *be a locally finite split graded semisimple Lie algebra. Then* $\mathfrak{g}^0$ *is a splitting Cartan subalgebra of* $\mathfrak{g}$. *In particular, every locally finite split graded semisimple Lie algebra is split.*

Proof. Extend $\mathfrak{g}$ by a subspace $\mathfrak{d} \subseteq \operatorname{Hom}(Q, \mathbb{K})$ such that $\mathfrak{g}_{\mathfrak{d}} = \mathfrak{g} \rtimes \mathfrak{d}$ is a split Lie algebra with splitting Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^0 \oplus \mathfrak{d}$. If $\mathfrak{g}$ is not contained in the $\mathfrak{h}$-invariant generalized Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}_{\mathfrak{d}}$, then $\mathfrak{r} \cap \mathfrak{g} \neq \{0\}$ where $\mathfrak{r}$ is the maximal locally solvable ideal of $\mathfrak{g}_{\mathfrak{d}}$. Hence $\mathfrak{r} \cap \mathfrak{g}$ is a non-trivial semisimple ideal of $\mathfrak{g}$. Since, according to [BaSt95a, Corollary 3.2], a locally finite semisimple Lie algebra is not locally solvable, this is a contradiction, showing that $\mathfrak{g} \subseteq \mathfrak{s}$. Therefore all roots of Δ are integrable, and the statement follows from Lemma IV.6. ∎

V. LOCALLY FINITE ROOT SYSTEMS

In this section we show that the root system of a locally finite split semisimple Lie algebra is the directed union of finite root subsystems of semisimple type, and, moreover, that the root system of a locally finite split simple Lie algebra is the directed union of finite root subsystems of simple type. As a consequence of the latter we see that a locally finite split simple Lie algebra is the directed union of finite dimensional simple subalgebras.

Definition V.1. (a) If Δ is the root system of a locally finite split semisimple, resp. simple, Lie algebra, then Δ is called a *locally finite root system of semisimple type*, resp. *simple type*, or simply a *locally finite root system*. If $\mathfrak{g}$ is finite dimensional, then Δ is called a *finite root system of semisimple*, resp. *simple*, *type*.

(b) Two locally finite root systems Δ and Δ' are said to be isomorphic if there exists a vector space isomorphism $\Psi\colon \operatorname{span}_{\mathbb{K}} \Delta \to \operatorname{span}_{\mathbb{K}} \Delta'$ such that $\Psi(\Delta) = \Delta'$. In this case, Ψ is called an *isomorphism of the root systems*.

In the following $(\mathfrak{g}, \mathfrak{h})$ denotes a locally finite split semisimple Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition. Note that, according to Theorem III.19, all roots of Δ are integrable and $\mathfrak{h} = \operatorname{span}_{\mathbb{K}} \check{\Delta}$.

Lemma V.2. *Each finite root subsystem Δ_0 of Δ is of semisimple type. In particular, Δ_0 is isomorphic to the root system of the finite dimensional semisimple Lie algebra $\mathfrak{g}_{\Delta_0}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\Delta_0}$, the restriction map $r\colon \operatorname{span}_{\mathbb{K}} \Delta_0 \to \operatorname{span}_{\mathbb{K}} \Delta(\mathfrak{g}_{\Delta_0}, \mathfrak{h}_{\Delta_0})$ being an isomorphism.*

Proof. If $\Delta_0 \subseteq \Delta$ is a finite root subsystem, then $\mathfrak{g}_{\Delta_0}$ is a finite dimensional semisimple separated Lie algebra and $\mathfrak{h}_{\Delta_0}$ is a Cartan subalgebra of $\mathfrak{g}_{\Delta_0}$ (Proposition III.7). Moreover the linear map r satisfies $r(\Delta_0) = \Delta(\mathfrak{g}_{\Delta_0}, \mathfrak{h}_{\Delta_0})$ (Lemma III.5(b)). Since $\mathfrak{g}_{\Delta_0}$ is the $\mathfrak{h}$-invariant Levi complement of the Lie algebra $\mathfrak{h} + \mathfrak{g}_{\Delta_0}$, Theorem III.16 implies that $\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{g}_{\Delta_0}) \oplus \mathfrak{h}_{\Delta_0}$. This shows that r is injective, so that r is an isomorphism of the root systems Δ_0 and $\Delta(\mathfrak{g}_{\Delta_0}, \mathfrak{h}_{\Delta_0})$. ∎

Definition V.3. A root subsystem Δ_0 of Δ is called *full* if $\Delta_0 = (\operatorname{span}_{\mathbb{K}} \Delta_0) \cap \Delta$. For a subset $M \subseteq \Delta$ we call $\Delta_M := (\operatorname{span}_{\mathbb{K}} M) \cap \Delta$ the full root subsystem of Δ generated by M. Note that a full root subsystem Δ_0 of Δ is a maximal proper full root subsystem if and only if $\operatorname{span}_{\mathbb{K}} \Delta_0$ is a hyperplane in $\operatorname{span}_{\mathbb{K}} \Delta$.

Proposition V.4. *If M is a finite subset of Δ, then Δ_M is a finite root system of semisimple type. Moreover, if M is irreducible, then also Δ_M is irreducible.*

Proof. Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ that is generated by the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in M$. Then $\mathfrak{g}_0 = \mathfrak{g}_{\Delta_0}$ for a finite root subsystem Δ_0 of Δ that satisfies $\operatorname{span}_{\mathbb{K}} \Delta_0 = \operatorname{span}_{\mathbb{K}} M$ and $\operatorname{span}_{\mathbb{K}} \check{\Delta}_0 = \operatorname{span}_{\mathbb{K}} \check{M}$ (Proposition III.9). In view of Lemma V.2, this shows that the restriction map $r\colon \operatorname{span}_{\mathbb{K}} M \to (\operatorname{span}_{\mathbb{K}} \check{M})^*$ is an isomorphism. Hence the inclusion

$$\Delta_M \subseteq \{\alpha \in \operatorname{span}_{\mathbb{K}} M : (\forall \beta \in M)\alpha(\check{\beta}) \in \mathbb{Z}, |\alpha(\check{\beta})| \leq 3\},$$

which holds by Corollary III.10, implies that $r(\Delta_M)$ and furthermore Δ_M is finite. Moreover, Δ_M is symmetric because Δ is symmetric, so that Δ_M is a finite root subsystem of semisimple type.

Suppose that M is irreducible, but that Δ_M is not (cf. Definition III.21). Then there exists a root $\beta \in \Delta_M$ such that $\beta(\check{M}) = \{0\}$, entailing that $\beta = 0$, which is absurd. Hence also Δ_M is irreducible. ∎

PROPOSITION V.5. *Denote by $F(\Delta)$ the finite and by $I(\Delta)$ the finite irreducible subsets of Δ.*

(a) *The root system Δ is the directed union of the finite root subsystems Δ_M, $M \in F(\Delta)$, of semisimple type. Moreover, if Δ is irreducible, then Δ is the directed union of the finite root subsystems Δ_M, $M \in I(\Delta)$, of simple type.*

In any case, $(\mathrm{span}_{\mathbb{K}} M) \cap \Delta = (\mathrm{span}_{\mathbb{Q}} M) \cap \Delta$ for each finite subset $M \subseteq \Delta$.

(b) *The Lie algebra $\mathfrak{g}$ is the directed union of finite dimensional semisimple subalgebras $\mathfrak{g}_{\Delta_M}$, $M \in F(\Delta)$. If, in addition, $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is the directed union of finite dimensional simple subalgebras $\mathfrak{g}_{\Delta_M}$, $M \in I(\Delta)$.*

Proof. (a) That Δ is the directed union of finite root systems of semisimple type follows immediately from Proposition V.4. If Δ is irreducible and M' is a finite subset of Δ, then any two roots $\alpha, \beta \in M'$ can be linked by a chain $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ of roots satisfying $\alpha_j(\check{\alpha}_{j+1}) \neq 0$ for $j = 1, \ldots, n-1$. Enlarging M' by all these chains leads to a finite irreducible subset M of Δ. Hence M' is contained in the finite irreducible root subsystem Δ_M of Δ, implying that Δ is the directed union of finite root systems of simple type.

The second part follows from the corresponding results for finite root systems of semisimple type.

(b) In view of Proposition III.7, the statement follows immediately from part (a). ∎

Remark V.6. Proposition V.5(b) can be used to prove the existence of a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (cf. [NeSt99]). ∎

VI. THE GENERALIZED BASE OF A LOCALLY FINITE ROOT SYSTEM

If Δ is an uncountable locally finite root system, then Δ does not contain a root base, i.e., Δ does not contain a linearly independent subset Φ with $\Delta \subseteq \mathrm{span}_{\mathbb{N}} \Phi \cup \mathrm{span}_{\mathbb{N}}(-\Phi)$ (cf. Remark VI.8). Nevertheless Δ contains a linearly independent subset Φ with the weakened requirement that $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Phi$. Such a subset is called a generalized base of Δ. In this section we prove that every locally finite root system has a generalized base.

DEFINITION VI.1. A subset Φ of a locally finite root system Δ that is linearly independent over $\mathbb{K}$ is called a *generalized base* if $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Phi$.

The following two lemmas about finite root systems of semisimple type will be crucial in the proof of the existence of a generalized base in Δ.

LEMMA VI.2. *Let Δ be a finite root system of semisimple type and Δ_0 a full root subsystem of Δ. Then Δ_0 is also a finite root system of semisimple type, and every root base Φ_0 of Δ_0 can be enlarged to a root base Φ of Δ. In particular, the Dynkin diagram of Δ contains the Dynkin diagram of Δ_0 as a subgraph.*

Proof. That Δ_0 is a finite root system of semisimple type is a well-known result of the finite dimensional theory and follows from the axiomatic characterization of a root system.

Now assume that Δ_0 is a maximal proper full root subsystem of Δ. Then $\operatorname{span}_{\mathbb{K}} \Delta_0$ is a hyperplane in $\operatorname{span}_{\mathbb{K}} \Delta$ and all roots that lie on one (fixed) side of this hyperplane form a parabolic system P of Δ which can be written as $P = \{\alpha \in \Delta : \alpha(Y) \geq 0\}$ for an element $Y \in \operatorname{span}_{\mathbb{Q}} \check{\Delta}$ with $Y \in (\Delta_0)^{\perp}$. If Δ_0^+ denotes the positive system of Δ_0 corresponding to the root base Φ_0, then $\Delta_0^+ = \{\alpha \in \Delta_0 : \alpha(X_0) > 0\}$ for an element $X_0 \in \operatorname{span}_{\mathbb{Q}} \check{\Delta}_0$ which can be chosen such that $|\alpha_1(X_0)| < \min\{|\alpha(Y)| : \alpha \in \Delta \backslash \Delta_0\}$ for all $\alpha_1 \in \Delta \backslash \Delta_0$. Then no root vanishes on the element $X := X_0 + Y$ and $\Delta^+ := \{\alpha \in \Delta : \alpha(X) > 0\}$ is a positive system of Δ with $\Delta_0^+ \subseteq \Delta^+ \subseteq P$. Moreover an indecomposible root of Δ_0^+ is also indecomposible in Δ^+, implying that the root base Φ_0 is contained in the set Φ of indecomposible roots in Δ^+ which is a root base of Δ. If Δ_0 is any full root subsystem of Δ, then there exists a chain of full root subsystems $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_n = \Delta$ such that Δ_j is a maximal proper full root subsystem of Δ_{j+1} for $j = 0, \ldots, n-1$. Thus a root base Φ_0 of Δ_0 can be enlarged to a root base Φ of Δ by applying the result for maximal proper full root subsystems successively.

Given the Dynkin diagram of the root system Δ with vertices labeled by the elements of Φ, we obtain the Dynkin diagram of the root system Δ_0 as the subgraph containing all vertices labeled by elements of Φ_0 and the corresponding edges. ∎

DEFINITION VI.3. Suppose Δ_0 is a maximal proper full root subsystem of a root system Δ, and let $f\colon \operatorname{span}_{\mathbb{Q}} \Delta \to \mathbb{Q}$ be a non-zero linear functional with $\ker f = \operatorname{span}_{\mathbb{Q}} \Delta_0$. Then for $c \in \mathbb{Q}$ the set $L_c := \{\alpha \in \Delta : f(\alpha) = c\}$ is called a *layer* of Δ with respect to Δ_0. A layer L_c is called *minimal* if $|c| = \min\{|c'| : L_{c'} \neq \emptyset, c' \neq 0\}$.

LEMMA VI.4. *Let Δ be a finite root system of semisimple type and Δ_0 a maximal proper full root subsystem of Δ. If $\gamma \in \Delta$ is an element in a minimal layer of Δ with respect to Δ_0, then $\Delta \subseteq \operatorname{span}_{\mathbb{Z}} \Delta_0 + \{n\gamma : n \in \mathbb{Z}, |n| \leq 6\}$.*

Proof. In view of Proposition V.5(a), there exists a non-zero linear functional $f\colon \operatorname{span}_{\mathbb{Q}} \Delta \to \mathbb{Q}$ such that $\ker f = \operatorname{span}_{\mathbb{Q}} \Delta_0$. Since γ is in a minimal layer of Δ with respect to Δ_0, we have $|f(\gamma)| = \min\{|f(\alpha)| : \alpha \in \Delta\backslash\Delta_0\}$, and we may assume w.l.o.g. that $f(\gamma) > 0$. Let Φ be a root base of Δ with the property that $\Phi_0 := \Phi \cap \Delta_0$ is a root base of Δ_0 (Lemma VI.2). Then there is only one root $\delta \in \Phi\backslash\Phi_0$ because $\mathrm{span}_{\mathbb{K}} \Delta_0$ is a hyperplane in $\mathrm{span}_{\mathbb{K}} \Delta$, and we may assume w.l.o.g. that $f(\delta) > 0$. Checking the tables Planche I–IX in the appendix of [Bou81] shows that each root $\alpha \in \Delta$ is expressible as a sum $\alpha = \sum_{\beta\in\Phi} n_\beta \beta$ where $n_\beta \in \mathbb{Z}$ with $|n_\beta| \le 6$. Hence

$$\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Phi_0 + \{n\delta : n \in \mathbb{Z}, |n| \le 6\}.$$

From $f(\alpha) = n_\delta f(\delta)$for $\alpha \in \Delta$ we derive $f(\delta) = \min\{f(\alpha) : \alpha \in \Delta, f(\alpha) > 0\} = f(\gamma)$ and further $\delta \in \gamma - \mathbb{N}_0[\Phi_0]$, which implies that $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_0 + \{n\gamma : n \in \mathbb{Z}, |n| \le 6\}$. ∎

In the following Δ denotes a locally finite root system.

LEMMA VI.5. *If Δ_0 is a maximal proper full root subsystem of Δ, then there exists a root $\gamma \in \Delta$ such that $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_0 + \mathbb{Z}\gamma$. In particular, every root $\gamma \in \Delta$ in a minimal layer of Δ with respect to Δ_0 satisfies $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_0 + \mathbb{Z}\gamma$.*

Proof. Fix a root $\delta \in \Delta\backslash\Delta_0$ and let $\alpha \in \Delta$. Since $\mathrm{span}_{\mathbb{K}} \Delta_0$ is a hyperplane in $\mathrm{span}_{\mathbb{K}} \Delta$, the root α is expressible as a linear combination $\alpha = \sum_{j=1}^n r_j \alpha_j + r_\alpha \delta$ where $\alpha_1, \dots, \alpha_n$ are linearly independent roots in Δ_0 and $r_1, \dots, r_n, r_\alpha \in \mathbb{K}$. Set $M_0 := \{\alpha_1, \dots, \alpha_n\}$ and $M := M_0 \cup \{\delta\}$. Then Δ_M is a finite root system of semisimple type, and Δ_{M_0} is a maximal proper full root subsystem of Δ_M. In view of Lemma VI.4, there exists a root $\gamma \in \Delta_M$ such that $\Delta_M \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_{M_0} + \{n\gamma : n \in \mathbb{Z}, |n| \le 6\}$. In particular, we have $\delta = \delta_0 + n\gamma$ where $\delta_0 \in \mathrm{span}_{\mathbb{Z}} \Delta_{M_0}$ and $n \in \mathbb{Z}$ with $|n| \le 6$. Inserting this sum in the expression for α yields $\alpha = \alpha' + (r_\alpha n)\gamma$ where $\alpha' \in \mathrm{span}_{\mathbb{K}} \Delta_{M_0}$ and $r_\alpha n \in \mathbb{Z}$ with $|r_\alpha n| \le 6$. From this we derive that $r_\alpha \in \{\frac{m}{n} : m, n \in \mathbb{Z}, n \ne 0, |m|, |n| \le 6\}$, and hence that the set $\{r_\alpha : \alpha \in \Delta\} \subseteq \mathbb{Q}$ is finite. This means that the root system Δ has a finite number of layers with respect to Δ_0. We can therefore find a root $\gamma \in \Delta$ in a minimal layer, for example, such that $r_\gamma = \min\{r_\alpha : \alpha \in \Delta, r_\alpha > 0\}$ (because Δ is symmetric).

For each root $\alpha \in \Delta$ there exists a finite set $M_0 \subseteq \Delta_0$ such that $\alpha = \alpha' + n_\gamma \gamma$ where $\alpha' \in \mathrm{span}_{\mathbb{K}} \Delta_{M_0}$ and $n_\gamma \in \mathbb{K}$. Since γ is also in a minimal layer of the root system $\Delta_M := (\mathrm{span}_{\mathbb{K}}(M_0 \cup \{\gamma\})) \cap \Delta$ with respect to Δ_{M_0}, we have $\alpha \in \Delta_M \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_{M_0} + \mathbb{Z}\gamma$ (Lemma VI.4). Hence $\Delta \subseteq \mathrm{span}_{\mathbb{Z}} \Delta_0 + \mathbb{Z}\gamma$. ∎

THEOREM VI.6. *The root system Δ contains a generalized base Φ. Moreover, if Φ is irreducible, then also Δ is irreducible.*

Proof. Let $\mathscr{M}$ be the non-empty set of pairs (Δ_1, Φ_1) where Δ_1 is a full root subsystem of Δ and Φ_1 is a generalized base of Δ_1. We define an ordering on $\mathscr{M}$ by declaring $(\Delta_1, \Phi_1) \leq (\Delta_2, \Phi_2)$ if $\Phi_1 \subseteq \Phi_2$. Then $(\mathscr{M}, \leq)$ is an inductively ordered set and hence contains a maximal element (Δ_0, Φ) by Zorn's Lemma. Suppose Δ_0 is strictly contained in Δ. Then Δ_0 is a proper full root subsystem of Δ, and, moreover, a maximal proper full root subsystem of some full root subsystem $\Delta' \subseteq \Delta$. By Lemma VI.5 there exists a root $\gamma \in \Delta'$ such that $\Delta' \subseteq \operatorname{span}_{\mathbb{Z}} \Delta_0 + \mathbb{Z}\gamma$, so that $(\Delta', \Phi \cup \{\gamma\})$ is an element of $\mathscr{M}$ greater than (Δ_0, Φ), which contradicts the maximality of (Δ_0, Φ). Hence $\Delta_0 = \Delta$ and Φ is a generalized base of Δ.

Suppose that Φ is irreducible, but that Δ is not. Then there exists a root $\beta \in \Delta$ such that $\beta(\check{\Phi}) = \{0\}$. Therefore $\beta = 0$ because $\operatorname{span}_{\mathbb{K}} \check{\Phi} = \operatorname{span}_{\mathbb{K}} \check{\Delta} = \mathfrak{h}$, which is absurd. ∎

The following lemma will be provided for references in [NeSt99]. It extends the result of Theorem VI.6 if Δ is of simple type.

LEMMA VI.7. *If the root system Δ is of simple type, then there exists an irreducible generalized base Φ of Δ with an element $\alpha_0 \in \Phi$ such that $\Phi \backslash \{\alpha_0\}$ is irreducible.*

Proof. Fix an element $\alpha_0 \in \Delta$, and let $\mathscr{M}$ be the non-empty set of pairs (Δ_1, Φ_1) where Δ_1 is a full root subsystem of Δ and Φ_1 is an irreducible generalized base of Δ_1 containing α_0 such that $\Phi_1 \backslash \{\alpha_0\}$ is also irreducible. As in the proof of Theorem VI.6, we define an ordering on $\mathscr{M}$ by declaring $(\Delta_1, \Phi_1) \leq (\Delta_2, \Phi_2)$ if $\Phi_1 \subseteq \Phi_2$. Then $(\mathscr{M}, \leq)$ is an inductively ordered set, and hence contains a maximal element (Δ_0, Φ) by Zorn's Lemma. Suppose Δ_0 is strictly contained in Δ. Then there exists a root $\gamma \in \Delta$ such that $\Phi' := \Phi \cup \{\gamma\}$ is an irreducible set consisting of linearly independent elements, so that Φ' is a generalized base of the full root subsystem $\Delta' := (\operatorname{span}_{\mathbb{K}} \Phi') \cap \Delta$. If $\Phi' \backslash \{\alpha_0\}$ is irreducible, then (Δ', Φ') is an element of $\mathscr{M}$ greater than (Δ_0, Φ), contradicting the maximality of (Δ_0, Φ). If, otherwise, $\Phi' \backslash \{\alpha_0\}$ is not irreducible, then the irreducibility of $\Phi \backslash \{\alpha_0\}$ yields $\gamma(\check{\beta}) = 0$ for all $\beta \in \Phi \backslash \{\alpha_0\}$. Moreover, there exists a root $\delta \in \Phi \backslash \{\alpha_0\}$ such that $\alpha_0(\check{\delta}) \neq 0$ and $\gamma(\check{\alpha}_0) \neq 0$ because Φ' is irreducible. >From the latter we derive, in view of Proposition I.7(iii), that there exists a root $\gamma' \in \{\gamma \pm \alpha_0\} \cap \Delta$, which satisfies $\gamma'(\check{\delta}) \neq 0$. Setting $\Phi'' := \Phi \cup \{\gamma'\}$ and $\Delta'' := (\operatorname{span}_{\mathbb{K}} \Phi'') \cap \Delta$, we obtain the element $(\Delta'', \Phi'') \in \mathscr{M}$ that is greater that (Δ_0, Φ), again contradicting the maximality of (Δ_0, Φ). We conclude that, contrary to our assumption, we have $\Delta_0 = \Delta$. ∎

Remark VI.8. If Δ is a countable root system of simple type, then Δ is the union of an ascending sequence of finite root subsystems of simple type, entailing, in view of Lemma VI.2, that Δ has a root base Φ. As in the finite dimensional case, a graph $\Gamma(\Phi)$ can be attached to Φ, the so-called generalized Dynkin diagram. Since any finite irreducible subset $M \subseteq \Phi$ is a generalized base of the finite root system Δ_M, which is of simple type, a finite connected subgraph of $\Gamma(\Phi)$ equals the Dynkin diagram of a finite dimensional simple Lie algebra. In particular, each vertex of $\Gamma(\Phi)$ meets at most three edges. These observations restrict the form of possible generalized Dynkin diagrams to five types, which all appear as generalized Dynkin diagrams of locally finite root systems of simple type (cf. [Sch60; MoPi95, Chap. 5.8]).

Suppose that Δ is uncountable and that Δ has a root base of Φ. Then, as we have seen above, each vertex in the generalized Dynkin diagram $\Gamma(\Phi)$ meets at most three edges. Therefore the number of vertices of $\Gamma(\Phi)$ is countable, implying that Δ is also countable, which contradicts the assumption. This shows that an uncountable locally finite root system has no root base.

PROPOSITION VI.9. *Let Φ be a generalized base of Δ. Then $\mathfrak{g}$ is generated by the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in \Phi$.*

Proof. Let $\beta \in \Delta$. Then β is expressible as a linear combination $\beta = \sum_{\alpha\in M} n_\alpha \alpha$ where M is a finite subset of Φ and $n_\alpha \in \mathbb{Z}\backslash\{0\}$ for $\alpha \in M$. Replacing some elements of M by their negative if necessary, we may assume w.l.o.g. that $n_\alpha \in \mathbb{N}$ for all $\alpha \in M$. Then we have $\beta = \sum_{j=1}^{n} \alpha_j$ where $\alpha_j \in M$ such that $\sum_{j=1}^{k} \alpha_j \in \Delta$ for $k = 1, \ldots, n$ by Lemma I.9. Hence $\mathfrak{g}^\beta = [\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_n}] \subseteq \langle\langle \mathfrak{g}(\alpha) : \alpha \in M \rangle\rangle$ follows from Proposition I.7(v). ∎

COROLLARY VI.10. *Let Φ be a generalized base of Δ. Then $\mathfrak{g}$ is the directed union of the finite dimensional semisimple subalgebras $\mathfrak{g}_{\Delta_M}$ where M is a finite subset of Φ.*

Proof. The finite dimensional semisimple subalgebras $\mathfrak{g}_{\Delta_M}$, where M is a finite subset of Δ, are obviously directed by inclusion. That their union equals $\mathfrak{g}$ follows from Proposition VI.9. ∎

VII. DIAGONAL AUTOMORPHISMS AND RATIONAL FORMS

In this short section the existence of a generalized base in the root system Δ of a locally finite split simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ is used to prove the existence of a $\mathbb{Q}$-form of $\mathfrak{g}$. As a means we introduce diagonal automorphisms, which also play an important role in proving the existence of a “compact” real form in a complex locally finite split semisimple Lie algebra in Section VIII.

DEFINITION VII.1. The automorphisms $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})_0 = \{D \in \mathrm{Aut}(\mathfrak{g}) : D|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\}$ are called *diagonal* automorphisms of $\mathfrak{g}$.

LEMMA VII.2. (a) *Let $D \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})_0$. Then for each $\alpha \in \Delta$ we have $D|_{\mathfrak{g}^\alpha} = \lambda_\alpha \,\mathrm{id}_{\mathfrak{g}^\alpha}$ where $\lambda_\alpha \in \mathbb{K}^*$ with $\lambda_\alpha \lambda_{-\alpha} = 1$.*

(b) *For an endomorphism* $D \in \mathrm{End}(\mathfrak{g})$ *with* $D|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ *and* $D|_{\mathfrak{g}^\alpha} = \lambda_\alpha \mathrm{id}_{\mathfrak{g}^\alpha}$ *where* $\lambda_\alpha \in \mathbb{K}^*$ *for* $\alpha \in \Delta$ *the following are equivalent*:

(1) *There exists a homomorphism of abelian groups* $f\colon (\mathrm{span}_{\mathbb{Z}} \Delta, +) \to (\mathbb{K}^*, \cdot)$ *such that* $f(\alpha) = \lambda_\alpha$ *for each root* $\alpha \in \Delta$.

(2) *We have* $D \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})_0$.

Proof. (a) Let $\alpha \in \Delta$ and $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$. Then for $h \in \mathfrak{h}$ we have

$$[h, D(x_\alpha)] = D([h, x_\alpha]) = \alpha(h)D(x_\alpha),$$

showing that $D(\mathfrak{g}^\alpha) = \mathfrak{g}^\alpha$. Moreover,

$$[x_\alpha, x_{-\alpha}] = D([x_\alpha, x_{-\alpha}]) = \lambda_\alpha\lambda_{-\alpha}[x_\alpha, x_{-\alpha}]$$

implies that $\lambda_\alpha\lambda_{-\alpha} = 1$.

(b) (1) $\Rightarrow$ (2): This implication follows from an easy calculation.

(2) $\Rightarrow$ (1): Let Φ be a generalized base of the root system Δ. Then the map $\Phi \to \mathbb{K}^*, \alpha \mapsto \lambda_\alpha$ extends uniquely to a homomorphism of abelian groups $f\colon \mathrm{span}_{\mathbb{Z}} \Delta \to \mathbb{K}^*$. In order to see that $f(\alpha) = \lambda_\alpha$ for all $\alpha \in \Delta$, consider the endomorphism D' of $\mathfrak{g}$ given by $D'|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $D'|_{\mathfrak{g}^\alpha} = f(\alpha)\,\mathrm{id}_{\mathfrak{g}^\alpha}$ for $\alpha \in \Delta$, which is an automorphism of $\mathfrak{g}$ by the implication (1)$\Rightarrow$(2). The automorphisms D and D' agree on the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in \Phi$, so that $D = D'$ follows from $\mathfrak{g} = \langle\langle \mathfrak{g}(\alpha) : \alpha \in \Phi \rangle\rangle$ (Proposition VI.9). ∎

DEFINITION VII.3. A $\mathbb{Q}$-subalgebra $\mathfrak{g}_{\mathbb{Q}}$ of a $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is called a $\mathbb{Q}$*-form* of $\mathfrak{g}$ if $\mathfrak{g}_{\mathbb{Q}}$ is a Lie algebra over $\mathbb{Q}$ and $\mathfrak{g} \cong \mathbb{K} \otimes \mathfrak{g}_{\mathbb{Q}}$. This means that a $\mathbb{Q}$-basis of $\mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{K}$-basis of $\mathfrak{g}$.

PROPOSITION VII.4. *Let* Φ *be a generalized base of* Δ, *and choose root vectors* $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ *such that* $[x_\alpha, x_{-\alpha}] = \check{\alpha}$ *for* $\alpha \in \Phi$. *Then the* $\mathbb{Q}$*-Lie algebra* $\mathfrak{g}_{\mathbb{Q}} = \langle\langle x_{\pm\alpha} : \alpha \in \Phi \rangle\rangle_{\mathbb{Q}}$ *generated by the elements* $x_{\pm\alpha}$ *for* $\alpha \in \Phi$ *is a* $\mathbb{Q}$*-form of* $\mathfrak{g}$.

Proof. It suffices to prove that for a finite subset $M \subseteq \Phi$ the $\mathbb{Q}$-Lie algebra $\langle\langle x_{\pm\alpha} : \alpha \in M \rangle\rangle_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of the Lie algebra $\mathfrak{g}_{\Delta_M}$ because $\mathfrak{g}$ is the directed union of the subalgebras $\mathfrak{g}_{\Delta_M}$ where M is a finite subset of Φ (Corollary VI.10). We may therefore assume w.l.o.g. that $\mathfrak{g}$ is finite dimensional. Then $\mathfrak{g}$ has a Chevalley basis $C = \{y_\alpha : \alpha \in \Delta\} \cup \{\check{\alpha}_1, \ldots, \check{\alpha}_l\}$ whose corresponding structure constants lie in $\mathbb{Z}$, showing that $\mathrm{span}_{\mathbb{Q}} C$ is a $\mathbb{Q}$-form of $\mathfrak{g}$ (cf. [Hum72, Sect. 25.2]). Since $[y_\alpha, y_{-\alpha}] = \check{\alpha}$ for $\alpha \in \Phi$, there exists a group homomorphism $f\colon \mathrm{span}_{\mathbb{Z}} \Delta \to \mathbb{K}^*$ such that $x_\alpha = f(\alpha)y_\alpha$ and $x_{-\alpha} = f(-\alpha)y_{-\alpha}$ for $\alpha \in \Phi$. Let $D_f \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})_0$ be the corresponding diagonal automorphism of $\mathfrak{g}$. Then $D_f(\mathrm{span}_{\mathbb{Q}} C)$ is a $\mathbb{Q}$-form of $\mathfrak{g}$ containing the elements $x_{\pm\alpha}$ for $\alpha \in \Phi$, showing that $\mathfrak{g}_{\mathbb{Q}} \subseteq D_f(\mathrm{span}_{\mathbb{Q}} C)$. By Proposition VI.9 we have $\mathfrak{g} = \langle\langle x_{\pm\alpha} : \alpha \in \Phi \rangle\rangle_{\mathbb{K}}$, showing that $\mathfrak{g}_{\mathbb{Q}}$ intersects every root space of $\mathfrak{g}$ non-trivially and hence that $\dim_{\mathbb{Q}} \mathfrak{g}_{\mathbb{Q}} \geq |\Delta| + |\Phi| = \dim_{\mathbb{K}} \mathfrak{g} = \dim_{\mathbb{Q}} D_f(\operatorname{span}_{\mathbb{Q}} C)$. Therefore $\mathfrak{g}_{\mathbb{Q}} = D_f(\operatorname{span}_{\mathbb{Q}} C)$, proving that $\mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $\mathfrak{g}$. ∎

COROLLARY VII.5. *The Lie algebra $\mathfrak{g}$ possesses a basis such that the resulting structure constants lie in $\mathbb{Q}$.*

VIII. COMPATIBLE INVOLUTIONS AND THE COMPACT REAL FORM

In this section we investigate involutions of a complex split Lie algebra that are in some sense compatible with the root decomposition. With their help, we extend the notion of a compact real form to complex locally finite split semisimple Lie algebras and prove that each such Lie algebra has a compact real form.

DEFINITION VIII.1. (a) Let $\mathfrak{g}$ be a complex Lie algebra and τ an involutive antilinear antiautomorphism of $\mathfrak{g}$. Then τ is called an *involution* of $\mathfrak{g}$ and the pair $(\mathfrak{g}, \tau)$ is called an *involutive Lie algebra*. Note that the involution τ determines a real form $\mathfrak{g}_{\mathbb{R}}^{\tau} := \{x \in \mathfrak{g} : \tau(x) = -x\}$ of $\mathfrak{g}$. Vice versa, if $\mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$, then the antilinear map τ of $\mathfrak{g}$ given by $\tau|_{\mathfrak{g}_{\mathbb{R}}} = -\operatorname{id}_{\mathfrak{g}_{\mathbb{R}}}$ is an involution of $\mathfrak{g}$ with $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}}^{\tau}$.

(b) Let $(\mathfrak{g}, \mathfrak{h})$ be a complex split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\alpha}$ the corresponding root decomposition. An involution τ of $\mathfrak{g}$ is said to be *compatible* with the root decomposition if $\tau(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha}$ for $\alpha \in \Delta$, and a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ is called *compatible* if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}}^{\tau}$ for a compatible involution τ. In this case, the triple $(\mathfrak{g}, \mathfrak{h}, \tau)$ (or $(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_{\mathbb{R}})$) is called an *involutive split Lie algebra*.

PROPOSITION VIII.2. *Let $(\mathfrak{g}, \mathfrak{h}, \tau)$ be an involutive split Lie algebra and Δ the corresponding root system. Let $\alpha \in \Delta_i$ and choose root vectors $0 \neq x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ with $[x_{\alpha}, x_{-\alpha}] = \check{\alpha}$. If we set $p = i(x_{\alpha} + \tau(x_{\alpha}))$, $q = x_{\alpha} - \tau(x_{\alpha})$, and $h = [x_{\alpha}, \tau(x_{\alpha})]$, then $\mathfrak{g}(\alpha)_{\mathbb{R}}^{\tau} := \mathfrak{g}(\alpha) \cap \mathfrak{g}_{\mathbb{R}}^{\tau} = \operatorname{span}_{\mathbb{R}}\{ih, p, q\}$ is a real form of $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$ and hence isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or to the compact Lie algebra $\mathfrak{su}(2)$. Moreover, one of the following cases occurs*:

(NS) *We have $\alpha(h) \in \mathbb{R}^{-}$ and $\tau(x_{\alpha}) \in \mu_{\alpha} x_{-\alpha}$ where $\mu_{\alpha} \in \mathbb{R}^{-}$. Then $\mathfrak{g}(\alpha)_{\mathbb{R}}^{\tau} \cong \mathfrak{sl}(2, \mathbb{R})$ (non-compact type).*

(CS) *We have $\alpha(h) \in \mathbb{R}^{+}$ and $\tau(x_{\alpha}) \in \mu_{\alpha} x_{-\alpha}$ where $\mu_{\alpha} \in \mathbb{R}^{+}$. Then $\mathfrak{g}(\alpha)_{\mathbb{R}}^{\tau} \cong \mathfrak{su}(2)$ (compact type).*

Proof. We have $\mathfrak{g}(\alpha) = (\mathfrak{g}(\alpha) \cap \mathfrak{g}_{\mathbb{R}}^{\tau}) \oplus (\mathfrak{g}(\alpha) \cap i\mathfrak{g}_{\mathbb{R}}^{\tau})$ because $\mathfrak{g}(\alpha)$ is invariant under the compatible involution τ and thus adapted to its eigenspace decomposition. Since the three dimensional Lie algebra $\mathfrak{g}(\alpha)_{\mathbb{R}}^{\tau}$

contains the linearly independent elements p, q, ih, it is spanned by them. From

$$\alpha(h)\tau(x) = -[h, \tau(x)] = -[\tau(h), \tau(x)] = \tau([h, x]) = \overline{\alpha(h)}\tau(x)$$

for $x \in \mathfrak{g}^\alpha$ we derive that $\alpha(h) \in \mathbb{R}$, so that $\alpha(h) > 0$ or $\alpha(h) < 0$ follows because the integrable root α does not vanish on $\mathbb{C}h = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$. Since $\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ and the root spaces are one dimensional, we have $\tau(x_\alpha) = \mu_\alpha x_{-\alpha}$ for some scalar $\mu_\alpha \in \mathbb{C}^*$. Therefore the equation $\alpha(h) = \alpha([x_\alpha, \tau(x_\alpha)) = 2\mu_\alpha$ shows that that $\alpha(h)$ and μ_α have the same sign. We may assume w.l.o.g. that $|\alpha(h)| = 2$. Then the relations $[q, p] = 2ih$, $[ih, p] = -\alpha(h)q$, and $[ih, q] = \alpha(h)p$ imply that $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{R})$ if $\alpha(h) = -2$ and $\mathfrak{g}(\alpha)_\mathbb{R} \cong \mathfrak{su}(2)$ if $\alpha(h) = 2$ (cf. [HiNe91, Bsp. III.1.16] or [Ne99, Example VII.2.1]). ∎

DEFINITION VIII.3. Let $(\mathfrak{g}, \mathfrak{h}, \tau)$ be an involutive split Lie algebra and Δ the corresponding root system. An integrable root $\alpha \in \Delta$ is called *non-compact* if $\mathfrak{g}(\alpha)^\tau_\mathbb{R} \cong \mathfrak{sl}(2, \mathbb{R})$ and *compact* if $\mathfrak{g}(\alpha)^\tau_\mathbb{R} \cong \mathfrak{su}(2)$.

DEFINITION VIII.4. Let $(\mathfrak{g}, \mathfrak{h}, \tau)$ be a locally finite involutive split semisimple Lie algebra and Δ the corresponding root system. The compatible involution τ is called *compact* if all roots of Δ are compact. In this case, $\mathfrak{g}^\tau_\mathbb{R}$ is called a compact real form. Note that for finite dimensional $\mathfrak{g}$ this definition is equivalent to the usual definition of a compact real form (cf. [HiNe91, Definition III.5.1]).

THEOREM VIII.5. *Let $(\mathfrak{g}, \mathfrak{h})$ be a complex locally finite split semisimple Lie algebra and Δ the corresponding root system. Let Φ be a generalized base of Δ, and choose for each $\alpha \in \Phi$ non-zero root vectors $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $[x_\alpha, x_{-\alpha}] = \check{\alpha}$. Then there exists a unique involution τ of $\mathfrak{g}$ such that $\tau(x_\alpha) = x_{-\alpha}$ for all $\alpha \in \Phi$. This involution is compatible and, moreover, compact.*

Proof. First we suppose that $\mathfrak{g}$ is finite dimensional. Then it is known from the finite dimensional theory that there exists a compact involution τ' of $\mathfrak{g}$. According to Proposition VIII.2 it satisfies $\tau'(x_\alpha) = \mu_\alpha x_{-\alpha}$ where $\mu_\alpha \in \mathbb{R}^+$ for $\alpha \in \Delta$. Let $D_f \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})_0$ be the diagonal automorphism of $\mathfrak{g}$ corresponding to the homomorphism $f\colon \mathrm{span}_\mathbb{Z}\,\Delta \to \mathbb{C}^*$ given by $f(\alpha) = (\mu_\alpha)^{1/2}$ for $\alpha \in \Phi$ (cf. Lemma VI.2). Then $\tau := D_f^{-1} \circ \tau' \circ D_f$ is a compatible involution of $\mathfrak{g}$ with $\tau(x_\alpha) = x_{-\alpha}$ for $\alpha \in \Phi$. Moreover, τ is compact because τ' is compact and $f(\Delta) \subseteq \mathbb{R}^+$. The uniqueness of τ follows because $\mathfrak{g}$ is generated by the test algebras $\mathfrak{g}(\alpha)$ for $\alpha \in \Phi$.

In the general case, if M is a finite subset of Φ, then M is a generalized base of the root system Δ_M. Therefore there exists a unique and compact involution τ_M of the Lie algebra $\mathfrak{g}_{\Delta_M}$ such that $\tau_M(x_\alpha) = x_{-\alpha}$ for $\alpha \in M$, and for finite sets $M_1, M_2 \subseteq \Phi$ with $\mathfrak{g}_{\Delta_{M_1}} \subseteq \mathfrak{g}_{\Delta_{M_2}}$ the involutions τ_{M_1} and

τ_{M_2} agree on $\mathfrak{g}_{\Delta_{M_1}}$. Since $\mathfrak{g}$ is the directed union of the subalgebras $\mathfrak{g}_{\Delta_M}$ for finite subsets $M \subseteq \Phi$ (Corollary VI.10), we can define a compatible involution τ on $\mathfrak{g}$ by setting $\tau(x) = \tau_M(x)$ for $x \in \mathfrak{g}(\Delta_M)$. This compatible involution satisfies $\tau(x_\alpha) = x_{-\alpha}$ for all $\alpha \in \Phi$ and is compact. Its uniqueness follows from Proposition VI.9. ∎

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